Linear Algebra
Exam 1
Fall 2021

October 14, 2021

Name: Solution Key

Honor Code Statement: I have neither given nor received unauthorized aid on this exam

Total possible: 70

Avg: $\frac{59}{70}$

S.D. $\frac{10.7}{70}$

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes or texts are not permitted. There is a two-hour time limit.

1. [5 points]The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that projects points onto the x_1 axis can be realized as a matrix transformation. What is the matrix transformation that achieves this?

We use Theorem 10 of Chapter 1: the standard matrix relies on knowing what T does to $\vec{e}_1$ and $\vec{e}_2$. From the given $T(\vec{e}_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$ and $T(\vec{e}_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$.

Thus, the standard matrix is $A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

2. [5 points] **Define** what it means for a set of vectors to be **linearly independent**. Give an example of a linearly independent set of vectors in $\mathbb{R}^3$ consisting of three vectors.

A set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is said to be linearly independent if the equation $x_1\vec{v}_1 + \ldots + x_p\vec{v}_p = \vec{0}$ has only the trivial solution.

A frequent error was to take Theorem 7 as the definition

Perhaps the simplest example of a linearly independent set of 3 vectors in $\mathbb{R}^3$ is

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

since $A = [\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$ has a pivot in each column

3. [3 points each] Each of the following statements is false. Amend each statement in as few words or symbols as possible to make a true statement, without simply negating the conclusion.

(a) The follow matrix is an example of an elementary matrix, $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$.

Possibilities to ammend: (by definition of elementary matrix)
- change the "-1" to 0
- change the "4" to 1

(b) If A is an $n \times n$ non-singular matrix, then the columns of A form a linearly dependent set.

Possibilities to ammend: (by Invertible Matrix Theorem)
- change non-singular to singular
- change linearly dependent to linearly independent

(c) If $AB = C$ and C has 3 rows and 2 columns, then A has 3 rows and B has 2 rows.

Possibilities to ammend: (by definition of matrix multiplication)
- change to "B has 2 columns"

4. [8 points] Find the inverse of the following matrix.

$$M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix}$$

The algorithm for finding the inverse follows from Theorem 7 of Chapter 2. We write:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & -2 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1/2 \\ 0 & 1 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Thus, $M^{-1} = \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 1/2 \\ -1 & 1 & 1 \end{bmatrix}$.

We may check $MM^{-1} = M^{-1}M = I_3$.

5. [5 points] Use the inverse from the ~~following~~ previous problem to solve the matrix equation $M\mathbf{x} = \mathbf{b}$, where b is the following column vector,

$$\mathbf{b} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}.$$

As $M^{-1} = \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 1/2 \\ -1 & 1 & 1 \end{bmatrix}$, we have that $\vec{x} = M^{-1}\vec{b}$ is

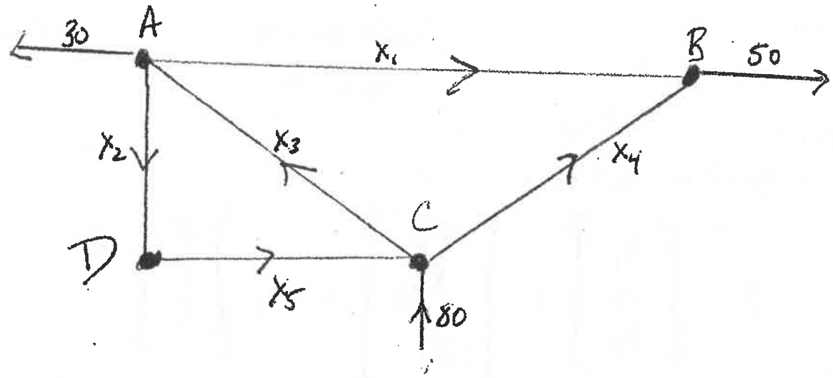
$$\begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 1/2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 11/2 \\ 1 \end{bmatrix}$$

6. [5 points] Find the following product MA, where M is given in a previous exercise and A is given below.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$MA = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 0 & -4 \\ 4 & 2 & 10 \end{bmatrix}$$

7. [5 points] Given the flow pattern of the network below, write a linear system that would allow you to determine the general flow pattern. (You don't need to solve the linear system.)



Note that at each node (vertex) we assume that flow is preserved. That is, "flow in" equals "flow out".

Thus, we obtain:

A: $x_3 = x_1 + x_2 + 30$

B: $x_1 + x_4 = 50$

C: $x_5 + 80 = x_3 + x_4$

D: $x_2 = x_5$

8. [10 points] The solution set of a homogeneous linear system in three variables is given. Write the solution set in parametric vector form. Give a geometric description of the solution set.

$$x_1 \quad -5x_3 = 0$$
$$x_2 - 8x_3 = 0$$
$$x_3 \textit{ is free}$$

The solution set is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ 8x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}.$$

The solution set is a line through the origin in the direction of the vector $\begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}$.

Suppose that a non-homogeneous linear system is given with the same coefficient matrix as this one (though note, I haven't given you the coefficient matrix). What can you say about the solution set of this system in relation to the one above?

If the system is consistent, then the solution set is parallel to the one given above. That is, it is a line not through the origin parallel to the vector $\begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}$. This follows from Theorem 6 of Chapter 1.

9. [4 points] Let A be an $n \times n$ matrix. The following statements are not all equivalent. There is one statement which can be removed and we'd be left with an equivalent set of statements. Which one is it? Why?

(a) A has fewer than n pivots.

(b) A is singular.

(c) The columns of A do not span $\mathbb{R}^n$.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has more than one solution.

(e) The columns of A do not form a linearly dependent set.

It is statement e. If statement e were to read "do form a linearly dependent set", then we'd have a set of equivalent statements by the Invertible Matrix Theorem. As such, we must remove it.

(Several people were unaware of the definition of singular.)

10. [8 points] **Fill in the four (4) blanks** Suppose that $CA = I_n$. Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

PROOF: If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $CA\mathbf{x} = C\mathbf{0} = \mathbf{0}$ and so $I_n\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$. This shows that the equation $A\vec{x} = \vec{0}$ has no free variables So every variable is a basic variable and every column of A is a pivot column.

11. [6 points] **Fill in the three (3) blanks** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

PROOF: Let $A = [\mathbf{v}_1 \cdots \mathbf{v}_p]$. Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns. If $p > n$, there are more variables than equations, so there must be a free variable. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.