Linear Algebra
Exam 1
Fall 2020

October 8, 2020

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.
Signature: C.F. Gauss

70 points

average 56 points

Directions: Complete all problems. Fill-in-the-blank problems are worth 2.5 points each and all others are worth 5 points each. Justify all answers/solutions. Calculators, notes or texts are not permitted. There is a two-hour time limit. Upon completion of the exam, please write the Honor Code Statement and give your signature.

1. The following vectors list the number of atoms of hydrogen (H), oxygen (O), calcium(Ca), and carbon(C) (in that order) for the following compounds:

$$H_3O: \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad CaCO_3: \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad H_2O: \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad Ca: \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad CO_2: \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Show how one could balance the following chemical equation by setting up an appropriate vector equation. You need NOT solve the equation.

$$H_3O + CaCO_3 \to H_2O + Ca + CO_2$$

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

where x_1 counts the # of H_3O, x_2 counts the # of $CaCO_3$, etc.

2. Find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}$$

We implement the algorithm provided in Theorem 7 of Chapter 2: We perform Gaussian Elimination on $[A \mid I_3]$.

pivots

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

$R2 \leftrightarrow R3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \end{array}\right]$$

$\frac{1}{2} R3$

$2R3 + R2$
$4R3 + R1$

Thus, $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}$.

One might check one's arithmetic by computing AA^{-1}, which, of course, equals I_3.

3. Use the inverse matrix A^{-1} that you found in the previous problem to solve the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. (That is, do not use Gaussian Elimination in finding a solution.)

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ 1/2 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

4. Write the matrix equation from the previous problem as a system of linear equations.

$$1x_1 \qquad - 4x_3 = 1$$

$$2x_3 = 2$$

$$x_2 - 2x_3 = -1$$

5. **Fill-in-the-blank** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. T is one-to-one if and only if A has ___n___ pivot columns.

By Theorem 12, T is one-to-one iff the columns of A are linearly independent iff A has n pivot columns.

6. **Fill-in-the-blank** Let A be an $n \times n$ matrix. Suppose that $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$. Then the equation $A\mathbf{x} = \mathbf{0}$ has ___more than the trivial solution___

7. Define what it means for a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ to be **onto**.

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if each $\vec{b} \in \mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$.

* Many folks failed to read the directions: a counter-example was sought.

** - A counterexample satisfies the hypothesis of the statement but not the conclusion.

8. [5 points each] Each of the following statements is false. For each statement give an example that illustrates the falsehood of the statement. Justify

(a) Any system of n linear equations in n variables has at most n solutions.

An example with no or one solution is not a counter-example!

$$x_1 - x_2 = 3$$
$$2x_1 - 2x_2 = 6$$

is a system of 2 equations in 2 unknowns with an infinite number of solutions since the two equations define the same line

(b) If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then the columns of A span $\mathbb{R}^m$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $A\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is consistent for $\vec{b} = [3]$. Yet, since A does not have a pivot in each row, the columns do not span $\mathbb{R}^2$.

(c) If an $m \times n$ matrix A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

then $\vec{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ are distinct solutions to $A\vec{x} = \vec{b}$

9. Consider the equation $A\mathbf{x} = \mathbf{b}$, where A is a 2×5 matrix with 2 pivot positions. What can you say about the number of solutions to this equation for a given $\mathbf{b} \in \mathbb{R}^2$?

As there are 2 pivot positions and 2 rows, the columns of A span $\mathbb{R}^2$: there will be a solution for each $\vec{b} \in \mathbb{R}^2$. With 5 columns, there will be 3 free variables and so infinite solutions for each $\vec{b} \in \mathbb{R}^2$.

10. Give~~n~~ an example of a 2×5 matrix with 2 pivot positions where the second column is NOT a linear combination of the other columns. Justify your solution.

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

It is obvious that there is no solution to

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

11. Show that the transformation T defined by $T(x_1, x_2) = (|x_2|, x_1)$, where $|y|$ denotes the absolute value of the real number y, is NOT a linear transformation.

Suppose $\vec{u} = (0,-1)$, $\vec{v} = (0,1)$.

Then $T(0,-1) = (1,0)$ and $T(0,1) = (1,0)$.

We also have $T(\vec{u}) + T(\vec{v}) = (2,0)$

However, $\vec{u} + \vec{v} = (0,0)$ and

$T(\vec{u}+\vec{v}) = T(0,0) = (0,0)$

Thus vector addition is not preserved. So T is not a linear transformation

One could also show that the transformation does not preserve scalar multiplication

12. Give the standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, where T first stretches/expands along the x_1-axis by a factor of 2, and then projects onto the x_1 axis.

We track what happens to $\vec{e}_1$ and $\vec{e}_2$.

Under stretching

$$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}2\\0\end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\1\end{bmatrix}.$$

Then under projection

$$\begin{bmatrix}2\\0\end{bmatrix} \to \begin{bmatrix}2\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\0\end{bmatrix}$$

Thus, $A = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$.

13. Let A be a 3×4 coefficient matrix. Suppose that the solution set to $A\mathbf{x} = \mathbf{0}$ is given in parametric vector form by $\mathbf{p} + s\mathbf{u} + t\mathbf{v}$, where $\mathbf{u} = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ and

$\mathbb{R}$

$s, t \in$ ~~mathbbR.~~ How many pivot columns does A have? Determine $\mathbf{p}$. Justify your answer.

The solution set of $A\vec{x} = \vec{0}$ always contains the trivial solution, i.e. $\vec{x} = \vec{0}$.

As a result of this knowledge, we can say that $\vec{p} = \vec{0}$. So the solution set can be more simply expressed as $s\vec{u} + t\vec{v}$ (which is a plane thru the origin).

This lets us also know there are 2 free variables in $A\vec{x} = \vec{0}$. As A is 3×4, it has 2 pivot columns.