Linear Algebra
Exam 1
Fall 2020

October 8, 2020

Name:
Honor Code Statement:
Signature:

Directions: Complete all problems. Fill-in-the-blank problems are worth 2.5 points each and all others are worth 5 points each. Justify all answers/solutions. Calculators, notes or texts are not permitted. There is a two-hour time limit. Upon completion of the exam, please write the Honor Code Statement and give your signature.

1. The following vectors list the number of atoms of hydrogen (H), oxygen (O), calcium(Ca), and carbon(C) (in that order) for the following compounds:

$$H_3O\colon \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad CaCO_3\colon \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad H_2O\colon \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad Ca\colon \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad CO_2\colon \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Show how one could balance the following chemical equation by setting up an appropriate vector equation. You need NOT solve the equation.

$$H_3O + CaCO_3 \rightarrow H_2O + Ca + CO_2$$

2. Find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}$$

3. Use the inverse matrix A^{-1} that you found in the previous problem to solve the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. (That is, do not use Gaussian Elimination in finding a solution.)

4. Write the matrix equation from the previous problem as a system of linear equations.

5. **Fill-in-the-blank** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. T is one-to-one if and only if A has ______________ pivot columns.

6. **Fill-in-the-blank** Let A be an $n \times n$ matrix. Suppose that $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$. Then the equation $A\mathbf{x} = \mathbf{0}$ has ____.

7. Define what it means for a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ to be **onto**.

8. [5 points each] Each of the following statements is false. For each statement give an example that illustrates the falsehood of the statement. Justify

   (a) Any system of n linear equations in n variables has at most n solutions.

   (b) If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then the columns of A span $\mathbb{R}^m$.

   (c) If an $m \times n$ matrix A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

9. Consider the equation $A\mathbf{x} = \mathbf{b}$, where A is a 2×5 matrix with 2 pivot positions. What can you say about the number of solutions to this equation for a given $\mathbf{b} \in \mathbb{R}^2$?

10. Give an example of a 2×5 matrix with 2 pivot positions where the second column is NOT a linear combination of the other columns. Justify your solution.

11. Show that the transformation T defined by $T(x_1, x_2) = (|x_2|, x_1)$, where $|y|$ denotes the absolute value of the real number y, is NOT a linear transformation.

12. Give the standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, where T first stretches/expands along the x_1-axis by a factor of 2, and then projects onto the x_1 axis.

13. Let A be a 3×4 coefficient matrix. Suppose that the solution set to $A\mathbf{x} = \mathbf{0}$ is given in parametric vector form by $\mathbf{p} + s\mathbf{u} + t\mathbf{v}$, where $\mathbf{u} = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ and $s, t \in \mathbb{R}$.

How many pivot columns does A have? Determine $\mathbf{p}$. Justify your answer.