Linear Algebra
Exam 1
Fall 2014

October 9, 2014

Name: Solution Key

Total: 70 points

Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Avg: 60.1

SD: 6.7

Directions: Complete all problems. Fill-in-the-blank problems are worth 2 points each and all others are worth 5 points each unless otherwise noted. Justify all answers/solutions. Calculators, notes or texts are not permitted. There is a two-hour time limit.

1. [10 points] Use Gaussian Elimination to solve the following linear system. Identify pivot columns. Identify free variables. Express your answer in parametric vector form.

$$2x_1 + 0x_2 - 6x_3 = -8$$
$$0x_1 + x_2 + 2x_3 = 3$$
$$3x_1 + 6x_2 - 2x_3 = -4$$

Note: Be sure to use proper notation. Please use $\sim$ for row equivalence, not $\rightarrow$.

We perform Gaussian Elimination on the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right] \underset{\frac{1}{2}R1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right] \underset{-3R1+R3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{array}\right] \underset{-6R2+R3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

pivot column $\qquad$ pivot column $\qquad$ pivot column.

$$\underset{-\frac{1}{5}R3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \underset{-2R3+R2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \underset{3R3+R1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Thus $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

As the solution is unique, there is no need to introduce a parameter.

Also note, there are no free variables: Columns 1, 2, and 3 are pivot columns.

2. [5 points] Rewrite the linear system from the previous question as both a vector equation and a matrix equation.

As a vector equation:

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} -6 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -4 \end{bmatrix}.$$

As a matrix equation:

$$\begin{bmatrix} 2 & 0 & -6 \\ 0 & 1 & 2 \\ 3 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -4 \end{bmatrix}$$

3. [5 points] A flop is one arithmetic operation. A flip is turning over one pancake. Suppose that the time it takes for a flop is the same as a flip. Which takes less time, flipping pancakes for each member of the student body (where each student gets 3 pancakes) or the forward phase of Gaussian Elimination on a 50×50-matrix?

According to the numerical note on page 20 of the text, the forward phase of Gaussian Elimination for a matrix of this size takes approximately $\frac{2}{3}(50)^3$ flops.

In regards to pancake flipping: first note each pancake gets flipped once and that there are roughly 2,500 students. Thus, total number of flips needed is $3 \cdot 2500 = 7500$.

Now note $7{,}500 < \frac{2}{3}(50)^3 = \frac{2}{3} \cdot 125{,}000$, and so I'd rather flip pancakes!

4. [5 points] Give an example of a consistent linear system with *more* equations than unknowns.

$$\begin{aligned} x_1 &= 2 \\ 5x_1 &= 10 \end{aligned}$$

One variable and two equations yielding a consistent system. Note that the 2nd equation is a scalar multiple of the first. The solution set is already given: $x_1 = 2$.

5. [5 points] Give a 3×3 matrix, not in echelon form, whose columns do *not* span $\mathbb{R}^3$. Justify your answer.

The following 3x3 matrix has columns that do not span $\mathbb{R}^3$, although it is in echelon form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. (Notice any vector in $\mathbb{R}^3$ w/ a non-zero 3rd entry will not be in the span.)

Now recall that elementary rows operations will not change this property. So let us be minimalist and perform one such operation to move the above matrix "out of" echelon form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix}.$$

7. [2 points] How do we denote the set of all linear combinations of the following vectors from $\mathbb{R}^n$, $\mathbf{v_1}, \ldots, \mathbf{v_p}$?

We denote this as Span $\{\vec{v}_1, \ldots, \vec{v}_p\}$.

8. [2 points] (Complete the sentence) The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

9. [2 points] A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$

has only the trivial solution.

10. [2 points] Give two vectors in $\mathbb{R}^4$ that form a linearly dependent set.

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \right\}$ forms a linearly dependent set - the second vector is a scalar multiple of the first.

See boxed note on page 58.

6. [3 points each] Each of the following statements is false. Amend each statement in as minimal a way as possible so as to create a true statement.

(a) If every column of an augmented matrix contains a pivot, then the corresponding system is consistent.

Replace augmented by coefficient. See Theorem 2 of Chapter 1.

(b) The vector $\mathbf{v}$ results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector $\mathbf{v}$.

Replace the first instance of $\vec{v}$ by $\vec{u}$.

(c) If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.

Remove the word not. See Theorem 4 of Chapter 1.

(d) If $\mathbf{x}$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.

Replace the word every by some.

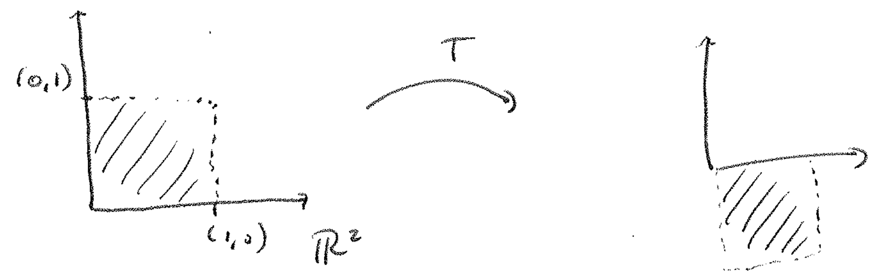
(e) A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of their inverses in the same order.

Replace the word same by reverse.

11. [5 points] Assume that T is a linear transformation. Find the standard matrix of T. T maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ by rotating points counter-clockwise about the origin through $3\pi/2$ radians.

To find the standard matrix, we must know about the image of $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under T.

First, a picture



T takes the unit square on the left to the unit square on the right.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By Theorem 10 of Chapter 1, we have

the standard matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, i.e.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}.$$

Sorry for my original typo - not present here!

12. [2 points for each blank] Let us prove the following theorem.

Theorem Any set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ is said to be linearly dependent if $p > n$.

PROOF: Let $A = [\mathbf{v_1}, \ldots, \mathbf{v_p}\}]$. Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of __n equations in p unknowns__. If $p > n$, there are more __unknowns / variables__ than equations, so there must be a __free variable__. Hence $A\mathbf{x} = \mathbf{0}$ has a __non-trivial solution__, and the columns of A are linearly dependent.□

This is Theorem 8 of Chapter 1.

13. [4 points] The following statement is false: If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are in $\mathbb{R}^3$ and $\mathbf{v_3}$ is *not* a linear combination of $\mathbf{v_1}, \mathbf{v_2}$, then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent. Provide a counterexample to show that it is false.

If $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$ and $\vec{v}_3$ is not a linear combination of $\vec{v}_1, \vec{v}_2$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

So here is a counterexample: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

14. [5 points] Prove that a linear transformation maps the zero vector to the zero vector.

We know by the definition that a linear transformation has the property $T(c\vec{u}) = cT(\vec{u})$ for any vector $\vec{u}$ and scalar c.

So, consider $T(\vec{0})$, which can be re-written as $T(0\vec{0})$. By the above property, this equals $0T(\vec{0})$.

Now note that the zero scalar times any vector results in the zero vector. That is, $0T(\vec{0}) = \vec{0}$. ☒

(Note: several students tried to use the standard matrix of T to run their argument. This is problematic since T may not map from $\mathbb{R}^n$ to $\mathbb{R}^m$.)