Linear Algebra
Exam 1
Fall 2014

October 9, 2014

Name:
Honor Code Statement:

Directions: Complete all problems. Fill-in-the-blank problems are worth 2 points each and all others are worth 5 points each unless otherwise noted. Justify all answers/solutions. Calculators, notes or texts are not permitted. There is a two-hour time limit.

1. [10 points] Use Gaussian Elimination to solve the following linear system. Identify pivot columns. Identify free variables. Express your answer in parametric vector form.

$$2x_1 + 0x_2 - 6x_3 = -8$$
$$0x_1 + x_2 + 2x_3 = 3$$
$$3x_1 + 6x_2 - 2x_3 = -4$$

2. [5 points] Rewrite the linear system from the previous question as both a vector equation and a matrix equation.

3. [5 points] A flop is one arithmetic operation. A flip is turning over one pancake. Suppose that the time it takes for a flop is the same as a flip. Which takes less time, flipping pancakes for each member of the student body (where each student gets 3 pancakes) or the forward phase of Gaussian Elimination on a 50×50-matrix.

4. [5 points] Give an example of a consistent linear system with *more* equations than unknowns.

5. [5 points] Give a 3×3 matrix, not in echelon form, whose columns do *not* span $\mathbb{R}^3$. Justify your answer.

6. [3 points each] Each of the following statements is false. Amend each statement in as minimal a way as possible so as to create a true statement.

   (a) If every column of an augmented matrix contains a pivot, then the corresponding system is consistent.

   (b) The vector $\mathbf{v}$ results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector $\mathbf{v}$.

   (c) If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.

   (d) If $\mathbf{x}$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.

   (e) A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of their inverses in the same order.

7. [2 points] How do we denote the set of all linear combinations of the following vectors from $\mathbb{R}^n$, $\mathbf{v_1}, \ldots, \mathbf{v_p}$?

8. [2 points] (Complete the sentence) The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the ____.

9. [2 points] A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

   ____

   has only the trivial solution.

10. [2 points] Give two vectors in $\mathbb{R}^4$ that form a linearly dependent set.

11. [5 points] Assume that T is a linear transformation. Find the standard matrix of T. T maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ by rotating points counter-clockwise about the origin through $3\pi/2$ radians.

12. [2 points for each blank] Let us prove the following theorem.

Theorem Any set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ is said to be linearly dependent if $p > n$.

PROOF: Let $A = [\mathbf{v_1}, \ldots, \mathbf{v_p}\}]$. Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of ____. If $p > n$, there are more ____ than equations, so there must be a ____. Hence $A\mathbf{x} = \mathbf{0}$ has a ____, and the columns of A are linearly dependent.$\square$

13. [4 points] The following statement is false: If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are in $\mathbb{R}^3$ and $\mathbf{v_3}$ is *not* a linear combination of $\mathbf{v_1}, \mathbf{v_2}$, then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent. Provide a counterexample to show that it is false.

14. [5 points] Prove that a linear transformation maps the zero vector to the zero vector.