Invertible Matrix Theorem

Theorem 1 *Let A be an $n \times n$ matrix. The following are equivalent:*

1. *A is an invertible (non-singular) matrix.*
2. *A is row equivalent to the $n \times n$ identity matrix.*
3. *A has n pivot positions.*
4. *The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.*
7. *The equation $A\mathbf{x} = \mathbf{b}$ has at least one (and in fact unique) solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*
8. *The columns of A span $\mathbb{R}^n$.*
9. *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.*
10. *There is an $n \times n$ matrix C such that $CA = I_n$.*
11. *There is an $n \times n$ matrix D such that $AD = I_n$.*
12. *A^T is an invertible matrix.*
13. *The determinant of A is not zero.*
14. *The columns of A form a basis of $\mathbb{R}^n$.*
15. *Col $A = \mathbb{R}^n$.*
16. *dim Col $A = n$.*
17. *rank $A = n$.*
18. *Nul $A = \{ \mathbf{0} \}$.*
19. *dim Nul $A = 0$.*
20. *The number 0 is not an eigenvalue of A.*
21. $(Col A)^{\perp} = \{\mathbf{0}\}$.
22. $(Nul A)^{\perp} = \mathbb{R}^n$.
23. *Row $A = \mathbb{R}^n$.*
24. *A has n nonzero singular values.*