Problem Sets

Graph Theory - MATH 247

May 17, 2023

Problem sets due at 5pm of the day indicated via the Canvas site. *Please note that the first printing of the text has some typographical errors. I'll try to alert these to you when I know about them.*

1. Due Wednesday, February 22
   Read: Syllabus, Thoughts on Homework
   Read: textbook's Preface and Prologue and Chapter 1
   Turn in (via Canvas site): Problems 2, 7, 9, 10, 12
   Note: In drawing the graphs from these problems, please draw such that structural characteristics are easily visible.

2. Due Friday, February 24
   Read: Chapter 2; Handout on Mathematical Induction; and the beginning of (p242-245, stopping at Corollary 1) "Weight Choosability of Graphs" by Bartnicki, Grytczuk and Niwczyk
   Turn in: Problems 2, 7, 8, 13, 19

   Also do: Problem 6, Prove Theorem 2.3 using induction on the number of distinct elements in the set

3. Due Friday, March 3
   Read: Chapter 3
   Turn in: Problems 5, 13, 17, 21, 23

   Install: software on a laptop so that you can use LaTeX. See the instructions on the course webpage. If this is not possible, please let me know. You will be asked to use LaTeX to typeset at least one problem from each future problem set, including this one.

   Also do: Problem 22; 25; and, Prove that every graph with no odd cycles is bipartite using induction on the number of vertices.

4. Due Friday, March 10

Read: Merlin's Magic Square by Don Pelletier and Noga Alon's proof of an upper bound on the domination number of the graph.

Turn in solutions to the following: (One may use mathematical software for any necessary matrix calculations done.) 1) In the game Merlin's Magic Square: if the winning pattern is given by $(1,1,1,0,0,0,1,1,1)$ and the initial pattern is $(0,1,0,1,0,1,0,1,0)$, find which buttons to press to win (and be sure to give the necessary supporting matrix calculation done). 2) Is there any initial pattern which requires that buttons 1–8 be pressed? Assume that the winning pattern is as suggested by the website we considered: (0,0,0,0,1,0,0,0,0). If yes, what is it? 3) Now turning to the game Light's Out (which has a slightly different rule set): consider the game on the 4-by-4 grid. Given any initial pattern and any winning pattern, is there always a way to win? 4) In Alon's proof, he makes use of the inequality $(1-p) < e^{(-p)}$. Prove this inequality.

Play here: https://aarongifford.com/magicsquare.html

Links to an external site.

Play here: https://www.logicgamesonline.com/lightsout/

5. Due Wednesday, March 15
   Read: Chapter 4
   Turn in: Problems 5, 10, 11(a),(c), 14. (At least one of these problems must be typeset using LaTeX.)

   Also do: 4, 11(b), 11(d), 15. Find the Prüfer code of a given labelled tree. Given a Prüfer code, draw the labelled tree.

6. Due Friday, March 31
   Read: Chapter 5
   Turn in: 2, 8, 13, 17

   Also do: Apply Dijkstra's algorithm to a weighted graph; know the definition of De Bruijn graphs and what the first few look like; State and prove a characterization of Eulerian digraphs (recall the definition of strongly-connected);

   This article about de Bruijn graphs and genome assembly is interesting:
   https://www.ncbi.nlm.nih.gov/pmc/articles/PMC5531759/

7. Due Friday, April 7
   Read: Chapter 6
   Turn in: 6, 9, 12, Prove Theorem 6.3
   Also do: 10; Prove that Theorem 6.2 is a corollary of Theorem 6.3; Characterize when the graph $K_{n_1,n_2,\ldots,n_p}$ is hamiltonian (and justify the characterization, of course!); Prove that the Petersen graph is not hamiltonian; Prove that Q_4 is hamiltonian (an

induction proof for Q_n?); Give an example of a graph on 6 vertices that shows that the greedy approach to TSP can be arbitrarily bad

8. Due Monday, April 17
   Read: Chapter 7
   Turn in: 1, 4, 9, 15 *(For problem 15: the first printing of the text doesn't include the edges u_2u_3 and v_2v_3, so please insert these.)*, and 16

9. Due Friday, April 28
   Read: Chapter 8 and https://www.quantamagazine.org/mathematicians-prove-ringels-graph-theory-conjecture-20200219 and Introduction to "A Proof of Ringel's Conjecture"
   Turn in: 1, 7 (for part (a) also consider P_6 and P_7), 11, 13
   Also do: 6

10. Due Friday, May 5
    Read: Chapter 10
    Turn in: 4, 7, 11, 15
    Also do: 5, 6, 8, 12, 17; Prove the upper bound for the crossing number of $K_{m,n}$ where "kilns" and "sheds" are each on their own line and the lines are intersecting (as Zarankiewicz did in 1954)

11. Due Monday, May 15
    Read: Chapter 11
    Turn in: 5,8,9,10
    Also do: 7, 12; Construct a graph G that is neither a complete graph nor an odd cycle but has a vertex ordering relative to which greedy coloring uses $\Delta(G)+1$ colors; Prove or disprove: For every graph G, $\chi(G) \leq n(G) - \alpha(G) + 1$.