Graph Theory - MATH 247

Exam 3

May 24, 2023

Name:
Honor Code Pledge:

Signature:

Directions: Complete 6 of 7 problems. There is a time limit of 3 hours. Best of luck!

Thanks for your work and attention throughout the semester. Peace, J.

1. For one of the problems from p-set 10, you had to find a planar graph for which every vertex had degree 5 or more. Many of you did this by providing a planar drawing of the icosahedron, a graph on 12 vertices (and one of the five Platonic solids, whether you realized it or not). Show that there is no graph on fewer than 12 vertices that does the same. That is, prove that every simple planar graph with fewer than 12 vertices has a vertex of degree at most 4. *Hint: edge bound.*

2. Give an embedding of K_7 on the torus (i.e. draw K_7 on the torus without crossings).

3. Prove that every k-chromatic graph with n vertices has at least $\binom{k}{2} = \frac{k(k-1)}{2}$ edges. [Writing limit of five sentences. Hint: consider an optimal coloring.]

4. Consider the graph formed taking C_5 and adding a vertex and making it adjacent to all other vertices. Call this graph W (for wheel). Determine the chromatic number of W. Also, if there exists an ordering of the vertices for which greedy coloring uses more than the optimal number of colors, then give it or explain why such an ordering doesn't exist.

5. Show that C_5 and the bow-tie graph (i.e. the 5-vertex graph formed by two triangles intersecting in a point) are **not** graceful.

6. **Steiner Triple Systems** For $n = 9$, there is a unique Steiner triple system, which is, in fact, a Kirkman system. One can form this triple system in the following way. Take the 9 points to be the *pairs* of elements from the integers modulo 3. The blocks are all sets of three distinct pairs having sum zero, i.e. sum $(0, 0)$. Prove that this construction does indeed yield a Steiner triple system. Next, show that it is a Kirkman system, i.e. show that the blocks can be grouped into parallel classes.

7. An embedding of a graph on a surface is regular if its faces all have the same length. Construct a regular embedding of $K_{3,3}$ on the torus. Before doing so, find the number and length of each face. *Hints: One result that follows from (Liza's favorite) First Theorem of Graph Theory is helpful; another that is helpful is Euler's Formula for S_γ, which states $n - e + f = 2 - 2\gamma$ holds for n-vertex graphs that embed on S_γ.*