Graph Theory - MATH 247

Exam 3

May 24, 2021

Name:
Honor Code Pledge:

Signature:

Directions: Complete 6 of 8 problems, including the Sudoku problem and one proof by induction. There is a time limit of 3 hours.

Best of luck!
Thanks for your work and attention throughout the semester. Peace, J.

1. Find all cycles C_n for $n \geq 3$ for which the graph complement $\overline{C}_n$ is non-planar.

2. Let P_n denote the path on n vertices. Determine the values of n for which there is a vertex ordering relative to which greedy coloring uses $\Delta(P_n)+1$ colors (whereas $\chi(P_n) = \Delta(P_n)$ for $n \geq 2$). For those values of n for which such an ordering exists, give an ordering that demonstrates this.

3. For what values of r, s is the graph $K_{r,s}$ planar? $K_{3,4}$ is not planar. What's the fewest number of edges that need to be deleted from $K_{3,4}$ to arrive at a planar graph? Why?

4. Euler's formula states that for a connected $n-$vertex plane graph, we have
$$n - e + f = 2.$$
Prove this by induction on the number of edges. (Hint: if every edge of such a graph is cut-edge, then the graph is tree. This implies that $e = n - 1$ and $f = 1$, in which case the formula holds. So, for the induction step, one need only consider graphs that are *not* trees.)

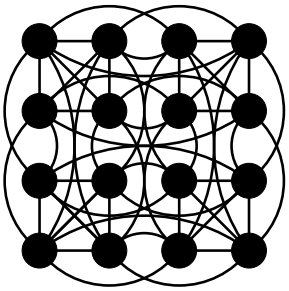


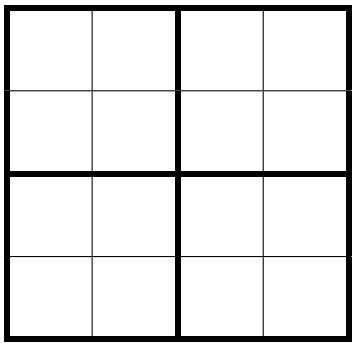
Figure 1: The 16-vertex 56-edge Shidoku graph

5. Prove that for every graph $\chi(G) \leq n(G) - \alpha(G) + 1$, where $n(G)$ denotes the number of vertices in G and $\alpha(G)$ denotes the size of the largest independent set.

6. Give a proper coloring of the cartesian product of C_5 and K_4 that uses the minimum number of colors.

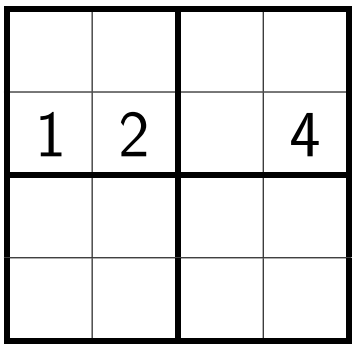
7. Prove Mantel's theorem by induction, where the induction step removes two adjacent vertices.

8. As you know, the following is a Shidoku board.



   The board and its rule set can be modeled by the graph SUD_2 in Figure 1.

   Suppose that we wish to apply the polynomial method to show that the following puzzle does not have a unique completion. Can you write down a polynomial of degree strictly less than 39 so that the nonzeros of the polynomial correspond to completions of the puzzle?



The redundancy graphs for the graph SUD_2 are as follows.

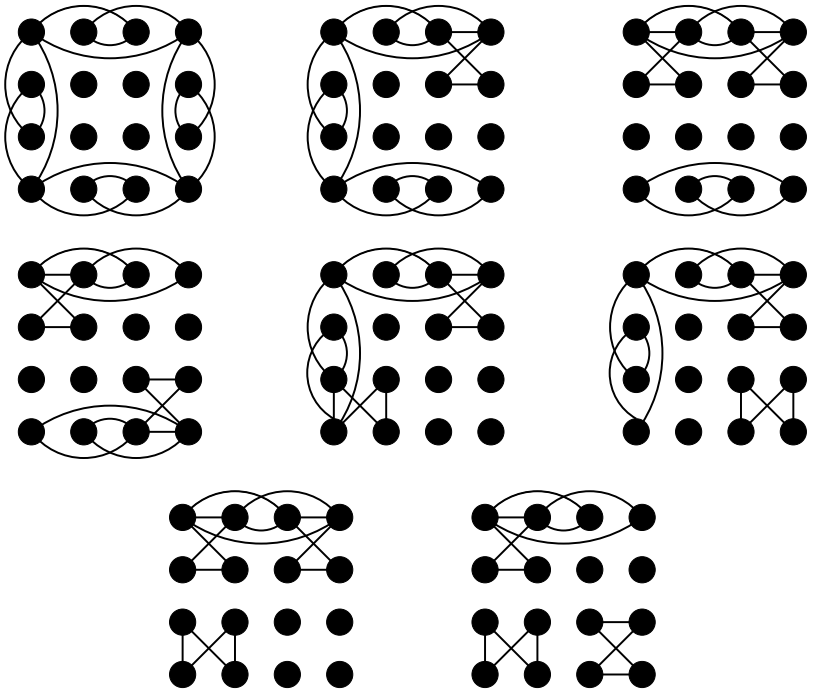


Figure 2: $G_1, \ldots, G_8$, from left to right, top to bottom