# Graph Theory - MATH 247

Exam 3

**Name:**
**Honor Code Pledge**

**Signature**

**Directions:** Please complete all problems. There is a time limit of 3 hours.

1. Given graphs $G$ and $H$, you proved $\chi(G \vee H) = \chi(G) + \chi(H)$. If $G$ and $H$ are color-critical is $G \vee H$ also color critical? If yes, prove it. If no, provide a counter-example.

2. Prove or disprove: If $G$ is a connected graph, then $\chi(G) \leq 1 + a(G)$, where $a(G)$ is the average of the vertex degrees.

3. Prove that every $k$-chromatic graph with $n$ vertices has at least $\binom{k}{2}$ edges.

4. We saw that neither of $K_5$ and $K_{3,3}$ are planar. If we delete an edge from either, the resulting graph is planar. However, we may draw $K_5$ and $K_{3,3}$ (not simultaneously) on the torus (a doughnut) without edge crossings. Show how this may be done. (Note that a torus may be formed from a sheet of paper by identifying the top side with the bottom and the left with the right. So you may think of "walking" off the top of the paper and "coming" out on the bottom of the paper, similarly with right and left. See the figure.)

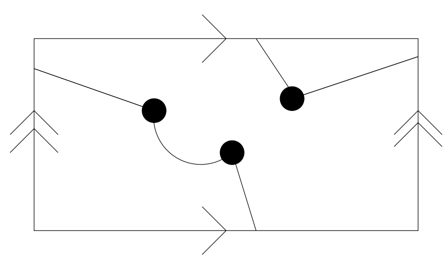

Figure 1: A funny way to draw $K_3$ on the torus

5. Let $G_{10}$ be the simple graph on 10 vertices, with vertices labelled $v_1, \ldots, v_{10}$. The edges of $G_{10}$ are formed by those vertices that are 3-close, that is, the edge set is $\{v_i v_j : |i-j| \leq 3\}$. Prove that $G_{10}$ is planar. (Hint: draw in a recursive fashion.) How many faces does $G_{10}$ have? How many faces would $G_{1,000}$ have? Is $G_{10}+(v_1 v_{10})$ planar?

6. Let $G$ be a simple graph, and let $H(G)$ be the graph with vertex set $V(G)$ such that $uv \in E(H)$ if and only if $u, v$ appear on a common cycle in $G$. Characterize the graphs $G$ such that $H$ is a complete graph.