Graph Theory - MATH 247

Exam 2

April 20, 2023

Name:
Honor Code Pledge:

Signature:

Directions: Please complete **five of six** problems. Each is worth 10 points. There is a time limit of 2 hours. No notes/texts/calculators/lap-tops/cell-phones/etc. allowed. Please remember to write and sign the Honor Code.

1. **Dijkstra's algorithm** Use Dijkstra's algorithm to find the shortest path between the vertices A and B in the graph below. Show the intermediary calculations that you would have made. Highlight the edges of the path that achieves this shortest path.

2. **Induction for a Hamilton circuit** The k-dimensional cube or *hypercube* Q_k is the simple graph whose vertices are the k-tuples with entries in $\{0, 1\}$ and whose edges are the pairs of k-tuples that differ in exactly one position. (So, Q_0 is a single point, Q_1 is a single edge incident to vertices labeled (0) and (1), Q_2 is isomorphic to the cycle on four vertices and Q_3 is what you would call a cube, with 8 vertices and 12 edges.) Use induction to show that Q_n is Hamiltonian for $n \geq 2$.

3. **Construct a non-Hamiltonian graph with prescribed degree condition** In one of the problems on one of the p-sets, you proved that if a graph G on $n \geq 2$ vertices has the property that for every pair of non-adjacent vertices u and v one has $deg(u)+deg(v) \geq n$, then G is Hamiltonian. Let's show that this statement cannot be improved. That is, let's show that there are graphs for which $deg(u)+deg(v) \geq n-1$ for each pair of non-adjacent vertices u and v and G is non-Hamiltonian. For example, can you give a graph on 4 vertices that is non-Hamiltonian and for any non-adjacent vertices the degree sum is at least 3? Start with $n = 2$, then $n = 3$ and so on.

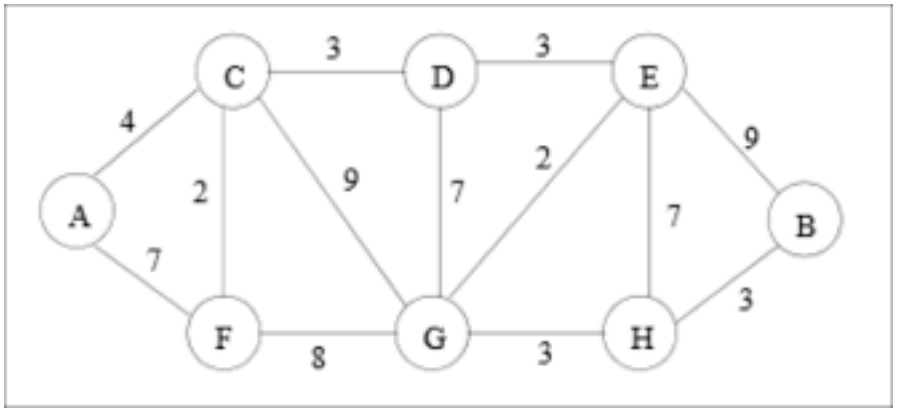
4. **Eulerian graphs**. We begin by defining the *line graph* of a graph G: For each edge in G, make a vertex in $L(G)$; for every two edges in G that have a vertex in common, make an edge between their corresponding vertices in $L(G)$. Show that the

line graph of the Petersen graph is Eulerian. (One needn't draw the graphs to answer this question. If you need a reminder of what the Petersen graph is, then I'll sell it to you for one point.)

5. **Hall's Theorem** Let G be a bipartite graph with bipartition given by X, Y. Suppose that we have the following condition that is weaker than Hall's condition: for each subset S of X we have $|N(S)| \geq |S| - 1$. Prove that G has a matching that saturates at least $|X| - 1$ vertices of X. (Incomplete) Hint: add a new vertex to ... and join it to ...

6. An n-dimensional De Bruijn graph of m symbols is a directed graph representing overlaps between sequences of symbols. The vertices consist of all possible length-n sequences of the given symbols; the same symbol may appear multiple times in a sequence. If one of the vertices can be expressed as another vertex by shifting all its symbols by one place to the left and adding a new symbol at the end of this vertex, then the latter has a directed edge to the former vertex.

   For the 3-dimensional De Bruijn graph on 3 symbols, using $\{a, b, c\}$ as the symbols, compute the following: the number of vertices of the graph, the degree of each vertex, the number of edges in the graph. Is the graph Eulerian?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