Graph Theory - MATH 247

Exam 2

November 14, 2024

Name:
Honor Code Pledge:

Signature:

Directions: Please complete **five of seven** problems. Each is worth 10 points. There is a time limit of 2 hours. No notes/texts/calculators/lap-tops/cell-phones/etc. allowed. Please remember to write and sign the Honor Code. Please put a slash through the two problems that will not be graded – that is, I will only grade the five of your choosing. Additional copies of the figures appears on the last page to help you with your thinking/problem-solving, and extra copies are available at the front of the room.

1. **Hamilton graphs** The graph in Figure 1 is not Hamiltonian. Prove this to be the case.

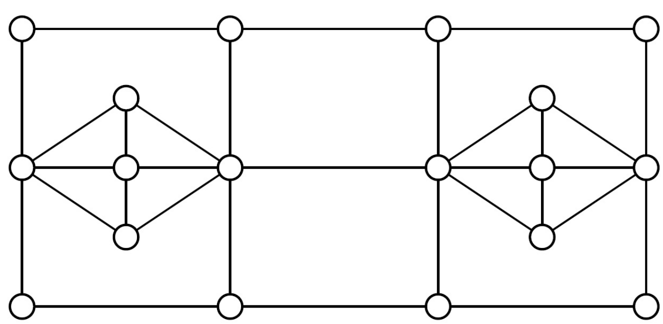

Figure 1: Show it's not Hamiltonian

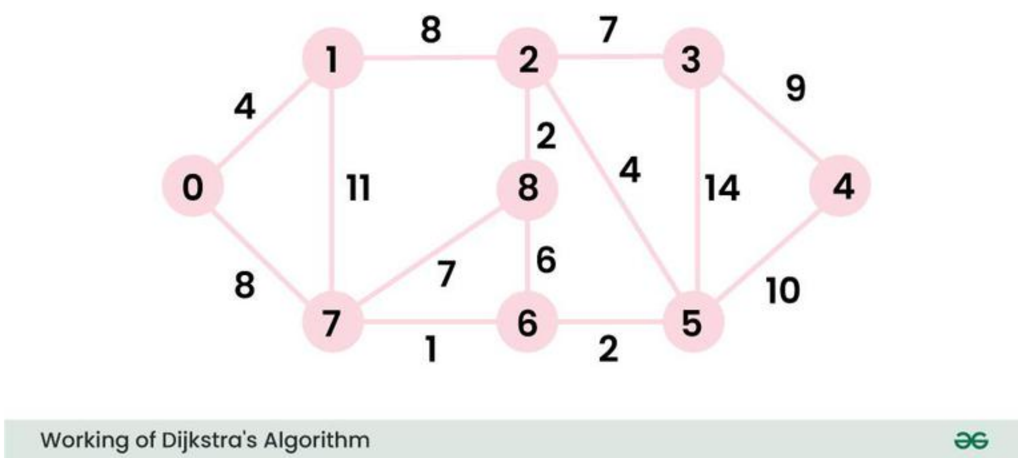

Figure 2: Compute shortest paths

2. **Dijkstra's algorithm** Use Dijkstra's algorithm to find the shortest path between the vertex 0 and all other vertices in the edge-weighted graph given in Figure 2, completing the table below. Show the **intermediary calculations** that you would have made. **Highlight the edges** of the minimum distance tree that is yielded by running the algorithm.

Vertex Label	Distance from Vertex 0
1	
2	
3	
4	
5	
6	
7	
8	

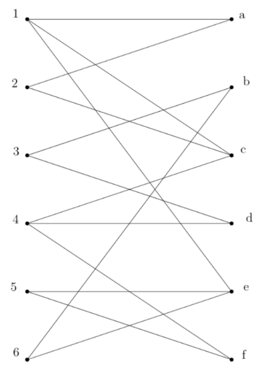

Figure 3: Matchings

3. **Eulerian graphs** The edge-weighted graph in Figure 2 is not Eulerian. Why not? (Be careful to reference the correct graph; that is, I'm asking about Figure 2, not Figure 3.) To make the graph Eulerian, we can duplicate some edges. How can we determine which edges to duplicate so as to minimize the total edge weight of an Eulerian circuit? (You needn't make the calculations.)

4. **The Petersen Graph**. Prove that the Petersen graph is *not* hamiltonian.

5. **Hall's Matching** [Writing limit: 5 sentences] One can check that the graph given in Figure 3 satisfies Hall's Matching Condition. If one does this directly, then how many things do you have to check? Please state the condition. Once you compute this number, you will notice that a considerable number of "checks" must be done. However, having the graph in front of you, what's a simple way to demonstrate that the Hall's Condition is satisfied.

6. **Ore's Theorem** I asked ChatGPT to give me a problem (and its solution) that I could ask you about the application of Ore's Theorem. What it wrote is given below. I have found mathematical and logical errors in what it wrote. Circle two of these errors and give a one sentence explanation of each error.

   Problem:

   Consider a simple, undirected graph G with 5 vertices. The degree sequence of the graph is $\{2, 3, 3, 4, 5\}$, meaning that the degrees of the vertices are 2, 3, 3, 4, and 5 respectively.

   Is the graph G Hamiltonian?

   Solution using Ore's Theorem:

   Ore's Theorem states that for a graph with $n \geq 3$ vertices, the graph is Hamiltonian if for every pair of non-adjacent vertices u and v,the sum of their degrees satisfies:

$$deg(u) + deg(v) \geq n.$$

   In this case, the graph has 5 vertices, so we need to check whether the sum of the degrees of every pair of non-adjacent vertices is at least 5.

   Let's check the degree sum for each pair of non-adjacent vertices.

   Vertex with degree 2: This vertex must be adjacent to two other vertices, so it can only be non-adjacent to the vertices with degrees 4 and 5.
   $2 + 4 = 6 \geq 5$ (satisfies)
   $2 + 5 = 7 \geq 5$ (satisfies)

   Vertex with degree 3: This vertex must be adjacent to two other vertices, so it can only be non-adjacent to the vertices with degrees 2 and 5. $3 + 2 = 5 \geq 5$ (satisfies)
   $3 + 5 = 8 \geq 5$ (satisfies)

   Another vertex with degree 3: Similar to the previous one, it can only be non-adjacent to the vertices with degrees 2 and 5.
   $3 + 2 = 5 \geq 5$ (satisfies)
   $3 + 5 = 8 \geq 5$ (satisfies)

   Vertex with degree 4: This vertex can only be non-adjacent to the vertices with degrees 2 and 3.

$4 + 2 = 6 \geq 5$ (satisfies)
$4 + 3 = 7 \geq 5$ (satisfies)

Vertex with degree 5: This vertex is adjacent to all other vertices, so there are no non-adjacent pairs involving this vertex.

Since for every pair of non-adjacent vertices, the sum of their degrees is at least 5, Ore's Theorem guarantees that the graph is Hamiltonian. Conclusion: *The graph is Hamiltonian.*

7. **Ore's Theorem** I asked ChatGPT to give me a problem (and its solution) that I could ask you about the application of Ore's Theorem. What it wrote is given below. I have found a fundamental **logical** error in what it wrote. Circle this error and give a one sentence explanation of the error.

Problem:

Consider a simple, undirected graph G with 6 vertices and the following degree sequence: $\{2, 3, 3, 4, 4, 5\}$. Is the graph G Hamiltonian?

Solution using Ore's Theorem:

We are given that G has 6 vertices, and the degree sequence is $\{2, 3, 3, 4, 4, 5\}$. We want to check if the graph is Hamiltonian using Ore's Theorem.

Ore's Theorem states that for a graph with $n \geq 3$ vertices, the graph is Hamiltonian if, for every pair of non-adjacent vertices u and v, the sum of their degrees satisfies:

$$deg(u) + deg(v) \geq n.$$

Since the graph has $n = 6$ vertices, Ore's Theorem tells us we need to check whether the degree sum of every pair of non-adjacent vertices is at least 6.

Step 1: Check Degree Sum for Non-Adjacent Pairs

We list the vertices and their degrees as follows:

$$deg(v_1) = 2,\ deg(v_2) = 3,\quad deg(v_3) = 3,\quad deg(v_4) = 4,\quad deg(v_5) = 4,\quad deg(v_6) = 5.$$

Now, we systematically check the degree sum for all pairs of non-adjacent vertices. We assume that the graph is not fully connected, so some of the vertices must be non-adjacent to each other. Checking Pairs:

Pair (v_1, v_2) $deg(v_1) + deg(v_2) = 2 + 3$. This pair fails because $5 < 6$.

Since we have already found a pair of non-adjacent vertices — (v_1, v_2)— where the sum of their degrees is less than 6, **Ore's Theorem guarantees that the graph is not Hamiltonian.**

Conclusion: The graph is not Hamiltonian because the degree sum condition fails for the pair (v_1, v_2) whose degree sum is 5 rather than the required 6 or more.

N.B. - I was trying to lead people to the logical error that is bold-faced. There are numerous errors beforehand, most notably with respect to the Handshaking Lemma.