Graph Theory - MATH 247 - Spring 2023

Exam 1

Name:
Honor Code Pledge:

Signature:

Directions: This exam is a closed-book, closed-notes exam. Cell phones should not be used at any time (even to check the time) - please put them away! Please complete **six of seven** problems. Each problem is worth 10 points. **Time limit:** 2 hours

1. **Prüfer codes:** Let $\mathbf{a} = (2, 2, 4, 4, 6, 6)$ be a Prüfer code. Construct the corresponding labelled tree. Also, given the labelled tree below, construct its Prüfer code.

2. **Prüfer codes:** Characterize those trees whose Prüfer codes have no repeated entries (i.e. the code is of length $n-2$ and the entries are distinct and come from $\{1, 2, \ldots, n\}$). Prove this characterization.

3. **A question based upon Theorem 2.3:** Construct/draw a graph of order 12 the degrees of whose vertices are precisely the integers from the following set $\{2, 3, 5, 7, 9, 11\}$. Illustrate the inductive step(s) of the proof of this theorem with what you draw.

4. **Lights out:** In the 4×1 game of Light's Out, prove whether or not every pattern is solvable (i.e. we can move from any patter to the "lights all out"). Prove that this is the case using linear algebra. Please give the necessary calculations.

5. Consider the graph obtained from K_5 by deleting two non-incident edges. Assign weights $\{1, 1, 2, 2, 3, 3, 4, 4\}$ to the edges in two ways: one way so that the minimum weight spanning tree is unique, and another way so that the minimum weight spanning tree is not unique.

6. **The greedy algorithm sometimes fails:** Prove that the greedy algorithm cannot guarantee a minimum weight spanning **path**. (The greedy algorithm would choose the cheapest available edge that maintains the possibility of a path.) Do this by constructing a graph on four vertices with only three distinct weights; show the path the greedy algorithm would return and show the minimum weight path. Can you generalize the construction to graphs on more than four vertices?

7. **Counter-intuitive?** 1) Prove that there are graphs for which deleting a vertex of minimum degree can reduce the average degree. 2) Prove that deleting a vertex of maximum degree cannot increase the average degree. *Caution:* Saying that this is "obvious" won't cut it. One should make some appropriate calculations.