Graph Theory - MATH 247 - Fall 2024

Exam 1

Name:
Honor Code Pledge:

Signature:

Directions: This exam is a closed-book, closed-notes exam. Cell phones should not be used at any time (even to check the time) - please put them away! Please complete **six of seven** problems. Each problem is worth 10 points. Some problems have writing limits - please observe them. **Time limit:** 2 hours. This exam is proctored by permission of the Dean of Faculty.

1. **Prüfer codes:** Let $\mathbf{a} = (1, 1, 3, 9, 3, 1)$. The code $\mathbf{a}$ is *not* a Prüfer code. Why not? (**Writing limit: two sentences.**) Also, given the labelled tree below, construct its Prüfer code.

2. **A question based upon Theorem 2.3:** Construct/draw a graph of order 10 the degrees of whose vertices are precisely the integers from the following set $\{1, 3, 5, 7, 9\}$. Illustrate the inductive step(s) of the proof of this theorem with what you draw.

3. [8 points] Determine the irregularity strength of $K_{2,2}$. [for 2 more points] Determine an upper bound for the irregularity strength of $K_{4,4}$.

4. Give the matrix equation that solves the Lights Out game on a 3-by-3 board where the initial pattern is to have the light in the upper-left corner and the light in the bottom-right corner on and all other lights off. You don't need to solve it. What must be true about the columns of the matrix that you write down in order for there to be a solution for any initial pattern?

5. Consider the following "statement" and its "proof". The statement is false and the proof is wrong. Give a graph G that shows that the statement is false. Indicate the error in the proof.

False Theorem: Every graph G with $|V(G)| \geq 3$ and all vertices of degree at least 2 contains a cycle of length 3.

False Proof: We proceed by induction on $|V(G)|$. As a base case, observe that the theorem is true when$|V(G)|$ since any simple graph on three vertices with all vertices of degree at least 2 must be a cycle of length 3.

To prove the inductive step, let G be a graph on $n-1$ vertices for which the theorem holds, and construct a new graph G' on n vertices by adding one new vertex to G and at least 2 edges incident with this new vertex. Since G contained a cycle of length 3, the graph G' also contains a cycle of length 3. This completes the proof.

6. Suppose that a graph has 20 vertices and 42 edges. Further suppose that each vertex has degree 4 or 5. Determine the number of vertices of each degree.

7. Let G be a tree with maximum degree 10. Prove that G has at least 10 leaves by using an extremal argument. To construct this argument, consider a vertex v of maximum degree and consider certain paths that begin at v.