Combinatorics - MATH 0345

Exam 3

May 16, 2024

Name:
Honor Code Pledge:

Signature:

Directions: Please complete five of the six questions. Electronic devices (including cell-phones), texts, and notes are not permitted in the exam room. There is a 3-hour time limit. Best of luck!

1. Consider the following 3-by-4 Latin rectangle A given below, with entries coming from $\mathbb{Z}_4$. Denote the entry in row i, column j by $a_{i,j}$, where $0 \leq i, j \leq 3$. Construct the 4-by-4 array B whose entry $b_{i,j}$ in position row i, column j satisfies

$$b_{i,j} = k, \quad \text{provided} \quad a_{k,j} = i$$

   and is blank otherwise. Next, in at most two sentences, say what specific property that A has that ensures that B has no repeated element in row 3.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix}$$

2. Count the number of Latin squares of order 4. Please use methods that we've derived throughout the class. That is, an *ad hoc* approach won't earn full points.

3. **Difference sets**. The following sets $\{0,1,3\}, \{2,6,7\}, \{5,10,12\}$ and $\{4,8,11\}$ contain entries from the integers modulo 13, i.e. are elements from $\mathbb{Z}_{13}$. They form a basis for a cyclic balanced incomplete block design. Compute the difference set for each of these. What property does the collection of these difference sets possess and why does this guarantee that the blocks generated cyclically form a BIBD? Determine the parameters (v, k, b, r, λ) of this design.

4. Compute the corner-symmetry group of a regular hexagon (the dihedral group D_6 of order 12).

5. Consider *just* the rotational symmetries of a regular hexagon, which forms the subgroup G of D_6. Let C denote the set of all red-blue-green colorings of the corners of the hexagon. Determine the number of nonequivalent colorings $N(G, C)$.

6. A *magic square* of order n is an an n-by-n array containing all the number from 1 to n^2 and such that each row and each column add up to the same number (which is called the *magic sum*). These can be constructed using a pair of orthogonal latin squares of order n. Construct a 5-by-5 magic square by adding together a pair A and B of orthogonal Latin squares of order 5 (with entries from 0 to 4), using the rule $5 \cdot a_{i,j} + b_{i,j} + 1$, doing the addition over the integers, not over $\mathbb{Z}_5$. What property guarantees that each row of the magic square has the same sum? What property guarantees that each number from 1 to n^2 occurs once? Derive a formula for the magic sum in the general case.