Combinatorics - MATH 0345

Exam 3

May 15, 2015

Name:
Honor Code Pledge:

Signature:

Directions: Please complete six of the seven questions. Electronic devices (including cell-phones), texts, and notes are not permitted in the exam room. There is a 3-hour time limit. Best of luck!

1. Solve the following recurrence relation by *using the method of generating functions*:

   $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}, (n \geq 3); h_0 = 1, h_1 = 1, h_2 = 2.$

2. Determine the generating function for the number h_n of non-negative integral solutions of

   $$2e_1 + 5e_2 + e_3 + 7e_4 = n.$$

   (*Hint:* define E_1 to be equal to $2e_1$. Note that you are not being asked to find a formula for h_n.)

3. Let us consider the 4−dimensional vector space over the finite field $\mathbb{F}_2$, denoted $\mathbb{F}_2^4$. (So, $\mathbb{F}_2$ consists of two elements, 0 and 1, and this set is imbued with two operations, addition and multiplication. We have the following facts: $0+1=1, 1+0=1, 1+1=0, 0+0=0$ and $0\times 1=0, 1\times 0=0, 1\times 1=1, 0\times 0=0$. The elements in $\mathbb{F}_2^4$ are $0,1$-vectors of length 4, and one may add two elements together in a component-wise fashion.) Let $\mathcal{X}$ be the set of all *nonzero* vectors of $\mathbb{F}_2^4$. Define a collection of blocks in the following manner: $\{\{\mathbf{x},\mathbf{y},\mathbf{z}\} | \mathbf{x}+\mathbf{y}+\mathbf{z}=\mathbf{0}, \mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{X}\}$. For instance, $\{(1,0,1,1),(1,0,0,0),(0,0,1,1)\}$ is a block since $(1,0,1,1)+(1,0,0,0)+(0,0,1,1)=(0,0,0,0)=\mathbf{0}$.

   Prove that the collection of blocks forms a balanced incomplete block design. Determine all the parameters of the design. That is, determine v, k, λ, r and b.

4. For each $n \geq 2$, construct a partial latin square of order n that has n cells filled and which **cannot** be completed to a latin square of order n.

5. Show that there are approximately $\frac{(n!)^2}{e}$ ways to construct the first two rows of a latin square of order n.

6. A *critical set* C in a latin square L or order n is a set

$$C = \{(i, j; k) | i, j, k \in \{1, 2, \ldots, n\}\}.$$

with the following two properties: (1) L is the only latin square of order n which has symbol k in cell (i, j) for each $(i, j; k) \in C$; and (2) no proper subset of C has property (1). A critical set is called *minimal* if it is a critical set of smallest possible cardinality for L.

Check that the following is a minimal critical set C for the latin square L. (Note: there are two things to prove here. You must prove that C is a critical set *and* that it is minimal.)

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & * & * \\ * & * & * & 3 \\ * & 4 & * & * \\ * & * & 2 & * \end{bmatrix}$$

7. The Middlebury Pranksters Frisbee Team have selected their five top players to compete against five members of the Nowhere Fools in a frisbee distance-throwing tournament.[1] They would like to construct a tournament with the following properties:

   (a) each player from the Pranksters throws against each player from the Fools;

   (b) there are no throw-offs/games between two people from the same team;

   (c) every person plays in each round;

   (d) every person plays at one of five locations exactly once during the tournament.

   Construct such a tournament for the Pranksters and Fools. Further, if we tally the number of throw-offs/games won by each team, what property of the tournament guarantees that there will be a winner, i.e. that draws are not possible?

[1] According to the World Flying Disc Federation, the world record distance for a disc-toss is 263.2 meters as set by Simon Lizotte of Germany in October 2014.