Combinatorics - MATH 0345

Exam 3

May 15, 2015

Name:
Honor Code Pledge:

Signature:

Directions: Please complete six of the seven questions. Electronic devices (including cell-phones), texts, and notes are not permitted in the exam room. There is a 3-hour time limit. Best of luck!

1. Solve the following recurrence relation by using the method of generating functions:

\[ h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}, \quad (n \geq 3); \quad h_0 = 1, h_1 = 1, h_2 = 2. \]

2. Determine the generating function for the number \( h_n \) of non-negative integral solutions of

\[ 2e_1 + 5e_2 + e_3 + 7e_4 = n. \]

(Hint: define \( E_1 \) to be equal to \( 2e_1 \). Note that you are not being asked to find a formula for \( h_n \).)

3. Let us consider the 4-dimensional vector space over the finite field \( \mathbb{F}_2 \), denoted \( \mathbb{F}_2^4 \).

(So, \( \mathbb{F}_2 \) consists of two elements, 0 and 1, and this set is imbued with two operations, addition and multiplication. We have the following facts: \( 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 0, 0 + 0 = 0 \) and \( 0 \times 1 = 0, 1 \times 0 = 0, 1 \times 1 = 1, 0 \times 0 = 0 \). The elements in \( \mathbb{F}_2^4 \) are \( 0, 1 \)-vectors of length 4, and one may add two elements together in a componentwise fashion.) Let \( \mathcal{X} \) be the set of all nonzero vectors of \( \mathbb{F}_2^4 \). Define a collection of blocks in the following manner: \( \{ \{ \mathbf{x}, \mathbf{y}, \mathbf{z} \} | \mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X} \} \). For instance, \( \{(1, 0, 1, 1), (1, 0, 0, 0), (0, 0, 1, 1)\} \) is a block since \( (1, 0, 1, 1) + (1, 0, 0, 0) + (0, 0, 1, 1) = (0, 0, 0, 0) = \mathbf{0} \).

Prove that the collection of blocks forms a balanced incomplete block design. Determine all the parameters of the design. That is, determine \( v, k, \lambda, r \) and \( b \).

4. For each \( n \geq 2 \), construct a partial latin square of order \( n \) that has \( n \) cells filled and which cannot be completed to a latin square of order \( n \).
5. Show that there are approximately \( \frac{(n!)^2}{e} \) ways to construct the first two rows of a latin square of order \( n \).

6. A **critical set** \( C \) in a latin square \( L \) or order \( n \) is a set

\[
C = \{(i, j; k) | i, j, k \in \{1, 2, \ldots, n\}\}.
\]

with the following two properties: (1) \( L \) is the only latin square of order \( n \) which has symbol \( k \) in cell \((i, j)\) for each \((i, j; k) \in C\); and (2) no proper subset of \( C \) has property (1). A critical set is called **minimal** if it is a critical set of smallest possible cardinality for \( L \).

Check that the following is a minimal critical set \( C \) for the latin square \( L \). (Note: there are two things to prove here. You must prove that \( C \) is a critical set and that it is minimal.)

\[
L = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 2 & * & * \\
* & * & * & 3 \\
* & 4 & * & * \\
* & * & 2 & * \\
\end{bmatrix}
\]

7. The Middlebury Pranksters Frisbee Team have selected their five top players to compete against five members of the Nowhere Fools in a frisbee distance-throwing tournament.\(^1\) They would like to construct a tournament with the following properties:

(a) each player from the Pranksters throws against each player from the Fools;
(b) there are no throw-offs/games between two people from the same team;
(c) every person plays in each round;
(d) every person plays at one of five locations exactly once during the tournament.

Construct such a tournament for the Pranksters and Fools. Further, if we tally the number of throw-offs/games won by each team, what property of the tournament guarantees that there will be a winner, i.e. that draws are not possible?

\(^1\)According to the World Flying Disc Federation, the world record distance for a disc-toss is 263.2 meters as set by Simon Lizotte of Germany in October 2014.