# Combinatorics - MATH 0345

Exam 3

December 14, 2021

**Name:**
**Honor Code Pledge:**

**Signature:**

**Directions:** Please complete six of the seven questions. Electronic devices (including cell-phones), texts, and notes are not permitted in the exam room. There is a 3-hour time limit. Best of luck!

1. Verify that the following three steps construct a Steiner triple system of index 1 with 13 varieties (using the elements of $\mathbb{Z}_{13}$).
   - Each of the integers $1, 3, 4, 9, 10, 12$ occurs exactly once as a difference of two integers in $B_1 = \{0, 1, 4\}$.
   - Each of the integers $2, 5, 6, 7, 8, 11$ occurs exactly once as a difference of two integers in $B_2 = \{0, 2, 7\}$.
   - The block $B_1$, the 12 blocks developed from $B_1$, the block $B_2$, and the 12 blocks developed from $B_2$ are the blocks of a Steiner triple system of index 1 with 13 varieties.

2. Let $n$ be an integer that counts the number of letters in your last name[1]. Give an example of an $(n-2) - by - n$ Latin rectangle that has precisely two completions. Prove that your example has this property.

3. [5 sentence limit] Let $L_1$ and $L_2$ be mutually orthogonal Latin squares of order $n$ on the set of symbols $\{0, \ldots, n-1\}$. Recall that a Latin square is said to be in *standard form* when $0, 1, \ldots, n-1$ occur in their natural order in the top-most row. Suppose that $L_1$ has this property, but that $L_2$ does not; $L_2$'s top-most row is $1, 2, \ldots, n-1, 0$. Give the permutation of the *elements* - not a permutation of columns (or rows) - that puts the square in standard form. Argue that this permutation of elements maintains the Latin property in both rows and columns. Argue that this permutation of elements maintains the property of orthogonality to $L_1$.

[1] Unless your first name is Abby, in which case $n = 7$.

4. Let us consider the set $\mathcal{C}$ of 3-colorings of the corners of the square. This time, however, we will restrict ourselves to a group *contained within* the dihedral group $D_4$ (i.e. just *some* of the motions). This *subgroup* $G$ consists of the following four elements: the identity $i$, the 180-degree rotation $\rho^2$, the reflection $\tau_1$ over a horizontal line and the reflection $\tau_2$ over a vertical line. Use Burnside's Theorem to compute the number $N(G, \mathcal{C})$ of nonequivalent colorings in $\mathcal{C}$ under $G$.

5. Write down the only symmetric, idempotent Latin square $L$ of order 3 (where the entries come from $\mathbb{Z}_3$) with rows labelled by $x$, where $0 \leq x \leq 2$ and columns labelled by $y$, where $0 \leq y \leq 2$. We will denote the entry in row $x$, column $y$ by $x \odot y$. We will use this square $L$ to construct a Steiner Triple System on the set $S$ of 9 ordered-pairs, $S = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$, so that $\lambda = 1$.

   Now write down the following triples:

   (a) All triples of the following form: $\{(x,0), (x,1), (x,2)\}$ for $0 \leq x \leq 2$.
   (b) All triples of the form $\{(x,0), (y,0), (x \odot y, 1)\}$ for $0 \leq x < y \leq 2$.
   (c) All triples of the form $\{(x,1), (y,1), (x \odot y, 2)\}$ for $0 \leq x < y \leq 2$.
   (d) All triples of the form $\{(x,2), (y,2), (x \odot y, 0)\}$ for $0 \leq x < y \leq 2$.

   To prove that these 12 blocks are indeed a Steiner Triple System, we need to show that each pair of ordered-pairs in $S$ occurs together once. How does $L$ being *idempotent* help in doing this? That is, which pair of order-pairs are guaranteed to exist together because the square is idempotent? (I don't wish to know about the cases where it doesn't help.)

6. [4 sentence limit] Recall that in a BIBD with $v$ varieties, $b$ blocks each of size $k$ and with each variety occurring $r$ times and each pair occurring together $\lambda$ times, we had established the following equalities:

$$bk = rv,$$
$$\lambda(v-1) = r(k-1).$$

   Fisher's inequality (which I'm not going to remind you of) along with the above two equalities can be used to bound the size $k$ of each block in terms of $v$ and $\lambda$. That is, give an upper bound on $k$ in terms of $\lambda$ and $v$ in a BIBD.

7. Consider a Steiner Triple system exists with parameters $b, v, k, r, \lambda$ and $k = 3$. Suppose that $\lambda = 6n + 1$. Prove that $v$ is congruent to 1 or 3 modulo 6.