Combinatorics - MATH 0345

Exam 3

December 11, 2012

Name:
Honor Code Pledge:

Signature:

Directions: Please complete all of the questions.

1. Prove that the n^{th} Fibonacci number f_n is the integer that is closest to the number
$$\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n.$$

2. Show that $B = \{0, 2, 3, 4, 8\}$ is a difference set in $\mathbb{Z}_{11}$. What are the parameters – give all five – of the SBIBD developed from B? Indicate how each of the parameters relates to the difference set – this is meant to keep you from writing out all of the blocks of the design and pointing to them and saying, "See, this is what I get."

3. Solve the following recurrence relation by using the method of generating functions: $h_n = h_{n-1} + h_{n-2}, (n \geq 2); h_0 = 1, h_1 = 3.$

4. Determine the generating function for the number f_n of solutions of the equation
$$e_1 + \ldots + e_k = n$$
in nonnegative *odd* integers $e_1, \ldots, e_k$. (Note that I am not asking you to find an explicit formula for f_n.)

5. Given a Steiner triple system of $\lambda = 1$, we say that the system is *resolvable* if we can partition the triples so that in each part we see each variety exactly once. Resolvable Steiner triple systems are also called *Kirkman triple systems*.

   Given a Kirkman triple system with 15 varieties, determine the number of parts in a partition of the above-mentioned type. Given a Kirkman triple system with $6n + 3$ varieties, determine the number of parts in a partition of the above-mentioned type. (That is, determine the number of resolvability classes that exist in such a design.)

6. A Latin square A based on $\mathbb{Z}_n$ of order n is *symmetric*, provided the entry a_{ij} at row i, column j equals the entry a_{ji} at column j, row i for all $i \neq j$. It is *idempotent* provided that its entries on the diagonal running from upper left to lower right are $0, 1, 2, \ldots, n-1$.

   Let $n = 2m + 1$, where m is a positive integer. Prove that the $n{-}\text{by}{-}n$ array whose entry a_{ij} in row i, column j satisfies

$$a_{ij} = (m+1) \times (i+j),$$

   (where arithmetic is done $mod(n)$) is a symmetric, idempotent, Latin square of order n. (*Remark:* The integer $m+1$ is the multiplicative inverse of 2 in $\mathbb{Z}_n$. Thus, our prescription for a_{ij} is to "average" i and j.) *Note:* There are three things to prove here!