Combinatorics
Exam 2
Spring 2024

April 18, 2024

Name:
Honor Code Pledge:

Signature:

Directions: Please complete **six of seven** of the problems.

1. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which no even integer is in its natural position.

2. How many functions from $\{1, 2, 3, 4, 5\}$ to $\{A, B, C\}$ are there? How many of these are onto functions?

3. **Combinatorial Reasoning** Give and prove a bijection between the subsets of the set of digits $\{0, 1, 2, \ldots, 9\}$ not containing two consecutive digits[1] and words of length ten that use the letters from the set $\{A, B\}$ in which there is a limit of 1 on consecutive A's. Then illustrate your bijection with a particular example or two.

4. **A corollary to PIE** Use a corollary to the Principle of Inclusion Exclusion to determine the number of integers between 1 and 1,000 that contain the digit 2 or contain the digit 3 or contain the digit 5. (Notice that this allows for the integer to contain one or more of these digits. For instance, the integer 230 is an integer we'd like to include in the counting.)

5. **Proof by Induction**. Let $\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$. Notice that ϕ satisfies the relationship $\phi^2 = \phi + 1$. Let f_n be the n^{th} Fibonacci number. Use induction and the relationship just stated to prove that $f_n \geq \phi^{n-2}$ for $n \geq 1$.

[1] So, the subset $\{1, 3, 7\}$ is included but $\{0, 3, 4, 5\}$ is not since $3, 4$ are consecutive.

6. **Recurrence relations** Solve the recurrence relation $h_n = 8h_{n-1} - 16h_{n-2} (n \geq 2)$ with initial values $h_0 = -1$ and $h_1 = 0$.

7. Consider the generating function $g(x) = \frac{1}{(1-x)^3}$. Compute the coefficient on the monomial x^8 in its expansion and give an interpretation of this coefficient.