Combinatorics - MATH 0345

Exam 2

April 16, 2015

Name:
Honor Code Pledge:

Signature:
Directions: Please complete six of seven problems.

1. [Brualdi] By integrating the binomial expansion, prove that, for a positive integer n,

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$$

2. [Brualdi] Let D_n denote the number of derangements of n objects. Prove that D_n is an even number if and only if n is an odd number.

3. [Brualdi] Solve the nonhomogeneous recurrence relation

$$h_n = 4h_{n-1} + 3 \times 2^n, \qquad (n \geq 1) \tag{1}$$
$$h_0 = 1. \tag{2}$$

4. [Brualdi] Let $f_0, f_1, \ldots, f_n, \ldots$ denote the Fibonacci sequence (where $f_0 = 0, f_1 = 1$). Use mathematical induction and the Fiobnacci recurrence to prove that

$$f_0 - f_1 + f_2 - \ldots (-1)^n f_n = (-1)^n f_{n-1} - 1.$$

5. Let k and n be positive integers. How many onto functions from $[k] = \{1, \ldots, k\}$ to $[n] = \{1, \ldots, n\}$ are possible? (Hint: it's as easy as PIE.)

6. Let us consider a function g of 13 variables, $x_1, \ldots, x_{13}$. Let $e_5(x)$ denote the 5^{th} elementary symmetric function

$$e_5(x_1, \ldots, x_{13}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_5 \leq 13} x_{i_1} x_{i_2} \cdots x_{i_5}.$$

(So, for any 5-subset of the 13 variables, there is an associated monomial using each variable precisely once and having coefficient 1. For example, $x_1x_4x_9x_{10}x_{11}$ is a term in the above sum.) Find the coefficient of the monomial $x_1 \ldots x_{13} = x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13}$ of the following polynomial.

$$g(x_1, \ldots, x_{13}) = (\sum_{i=1}^{13} x_i)^3 (e_5(x_1, \ldots, x_{13}))^2.$$

7. [Brualdi] Let p and s be positive integers. Give a combinatorial proof of the following identity:

$$\binom{p+1}{2} 2^{p-1} = \sum_{s=1}^{p} s^2 \binom{p}{s}.$$