1. Prove that
\[ \sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1, n_2, n_3, n_4} (-1)^{n_1-n_2+n_3-n_4} = 0, \]
where the summation extends over all nonnegative integral solutions of \(n_1 + n_2 + n_3 + n_4 = n\).

2. Determine the number of permutations of \(\{1, 2, \ldots, 8\}\) in which exactly four integers are in their natural positions.

3. Prove that the Fibonacci sequence is the solution of the recurrence relation
\[ a_n = 5a_{n-4} + 3a_{n-5}, \quad (n \geq 5), \]
where \(a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2,\) and \(a_4 = 3\). Then use this formula to show that the Fibonacci numbers satisfy the condition that \(f_n\) is divisible by 5 if and only if \(n\) is divisible by 5.

4. Let \(n\) and \(k\) be positive integers. Give a combinatorial proof of the following identity,
\[ n(n + 1)2^{n-2} = \sum_{k=1}^{n} k^3 \binom{n}{k}. \]
5. Let $f_0, f_1, f_2, \ldots$ denote the Fibonacci sequence. Evaluate the following expression for small values of $n$, conjecture a general formula and then prove it, using mathematical induction and the Fibonacci recurrence:

$$f_0^2 + f_1^2 + \ldots + f_n^2.$$

6. Use Newton’s binomial theorem to approximate $\sqrt{23}$. 