Combinatorics - MATH 0345

Exam 2

April 16, 2009

Name:
Honor Code Pledge:

Signature:

Directions: Please complete all but 1 problem. If you complete all six problems, I will count your best five.

1. Prove that

$$\sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1\ n_2\ n_3\ n_4} (-1)^{n_1-n_2+n_3-n_4} = 0, \tag{1}$$

   where the summation extends over all nonnegative integral solutions of $n_1 + n_2 + n_3 + n_4 = n$.

2. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which exactly four integers are in their natural positions.

3. Prove that the Fibonacci sequence is the solution of the recurrence relation

$$a_n = 5a_{n-4} + 3a_{n-5}, \quad (n \geq 5),$$

   where $a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2$, and $a_4 = 3$. Then use this formula to show that the Fibonacci numbers satisfy the condition that f_n is divisible by 5 if and only if n is divisible by 5.

4. Let n and k be positive integers. Give a combinatorial proof of the following identity,

$$n(n+1)2^{n-2} = \Sigma_{k=1}^{n} k^2 \binom{n}{k}.$$

5. Let $f_0, f_1, f_2, \ldots$ denote the Fibonacci sequence. Evaluate the following expression for small values of n, conjecture a general formula and then prove it, using mathematical induction and the Fibonacci recurrence:

$$f_0^2 + f_1^2 + \ldots + f_n^2.$$

6. Use Newton's binomial theorem to approximate $\sqrt{23}$.