Combinatorics
Exam 2
Fall 2021

November 11, 2021

Name:
Honor Code Pledge:

Signature:

Directions: Please complete **six of seven** of the problems.

1. When logging into a website for the first time, I frequently am faced with having to create a password meeting the specifications of the website. One such website asked for an 8-character password[1] that could use only the letters A, B, C, D, E and had to use each of these letters at least once. How many such passwords are there?

2. Below is a 5×5 chessboard upon which we'd like to place 5 non-attacking rooks. However, there are some forbidden positions (indicated in the color maroon) on the board where we cannot place rooks. How many such non-attacking rook placements are there?

3. Padovan sequence a_n begins as follows: $a_0 = 1, a_1 = 0, a_2 = 0$. The recurrence is given by $a_n = a_{n-2} + a_{n-3}$ for $n \geq 3$. For $n \geq 3$,, we define a_n as the number of compositions of n into parts that are odd and each part is 3 or greater. Examples: $a_{10} = 3$ counts 3+7, 5+5, 7+3; $a_6 = 1$ since $3 + 3 = 6$; $a_7 = 1$ since $7 = 7$. See the table below for other values.

   Find the generating function for this sequence. (I'm not asking you to find a formula for the n^{th} term, I'm just asking you to find $g(x)$ as a rational function, i.e. a quotient of two polynomials.)

4. Solve the following recurrence relation by examining the first few values for a formula and then proving your conjectured formula by induction.

[1] My favorite 8-character password is "Snow White and the Seven Dwarves". Ha! Ha! Hysterical.

$$h_n = -h_{n-1} + 1, (n \geq 1); h_0 = 0.$$

5. Let Q_n denote the number of permutations of $\{1, 2, \ldots, n\}$ in which none of the patterns $12, 23, \ldots, (n-1)n$ occurs. Show[2] that

$$Q_n = (n-1)!(n - \frac{n-1}{1!} + \frac{n-2}{2!} - \frac{n-3}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}).$$

6. Solve the non-homogeneous recurrence relation $h_n = 4h_{n-1} + 3 \times 2^n,\ (n \geq 1)$ and $h_0 = 1$.

7. Verify the recurrence relation given for the Padovan sequence for $n \geq 3$. That is, give a combinatorial argument as to why $a_n = a_{n-2} + a_{n-3}$ based upon the definition of a_n.

n	a_n
0	1
1	0
2	0
3	1
4	0
5	1
6	1
7	1
8	2
9	2
10	3
11	4
12	5
13	7
14	9

Table 1: The first few terms of the Padovan sequence

[2]It has been previously shown that $Q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n-1}{n-1}1!$. I'm not saying this is necessary, but I told Abby that you didn't have to memorize formulas so I'm keeping good on my promise, I hope.