Combinatorics - MATH 0345

Exam 2

November 11, 2010

Name:
Honor Code Pledge:

Signature:
Directions: Please complete all of the problems.

1. A talk show host has just bought 10 new jokes. Each night he tells some of the jokes. What is the largest number of nights on which you can tune in so that you never hear on one night at least all the jokes you heard on one of the other nights? (Thus, for instance, it is acceptable that you hear jokes 1, 2, and 3 on one night, jokes 3 and 4 on another, and jokes 1,2, and 4 on a third. It is not acceptable that you hear jokes 1 and 2 on one night and joke 2 on another night.)

2. At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that at least two of the gentlemen receive their own hats?

3. Solve the nonhomogeneous recurrence relation

$$h_n = 4h_{n-1} + 3 \times 2^n, \quad (n \geq 1)$$
$$h_0 = 1.$$

4. Let S be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3, \infty \cdot e_4\}$. Determine the generating function for the sequence $h_0, h_1, \ldots, h_n, \ldots$, where h_n is the number of n-combinations of S with the restriction that each e_i occurs a multiple-of-3 number of times.

5. Use the Multinomial Theorem to prove that

$$t^n = \Sigma \binom{n}{n_1, n_2, \cdots, n_t} \tag{1}$$

where the summation extends over all nonnegative integral solutions $n_1, n_2, \ldots, n_t$ of $n_1 + n_2 + \ldots + n_t = n$. (No credit given if you prove this in any other method than indicated.)

6. Let n be a positive integer and let $p_1, p_2, \ldots, p_k$ be all the different prime numbers that divide n. Consider the Euler function ϕ defined by

$$\phi(n) = |\{k : 1 \leq k \leq n, GCD\{k, n\} = 1\}|.$$

(For example, $\phi(6) = 2$ since the only integers relatively prime to and less than 6 are 1 and 5.)

Use the inclusion-exclusion principle to show that

$$\phi(n) = n\Pi_{i=1}^{k}(1 - \frac{1}{p_i}).$$

(And see that for $n = 6$ we have $p_1 = 2, p_2 = 3$ and so $\phi(6)$ is given by $6(1-\frac{1}{2})(1-\frac{1}{3}) = 2$.)