Combinatorics - MATH 0345

Exam 1

March 14 (π-day), 2024 $= \binom{21+3}{3}$, the twenty-second tetrahedral number

Name:
Honor Code Pledge:

Signature:

Directions: Please complete **six of the seven** problems. Justify all solutions — partial work receives partial credit. Each problem is worth ten points. Answers may be left in terms of factorials, binomial coefficients, products and sums of these, and the like. Some problems have a writing limit – the limit is indicated at the beginning of the problem. There is a time limit of 2 hours.

1. There's a group of 12 students. I have a round table that seats 4 students. The students that don't sit at the table will simultaneously run a marathon, with only 3 medals awarded (gold, silver and bronze).

   (a) How many ways are there for a simultaneous seating and marathon podium result to occur? (Ties in the marathon are not possible.)

   (b) How many ways are this to occur if we are also to satisfy the wish that a particular two students refuse to sit next to each other at the table and conspire to never simultaneously "podium" in the marathon?

2. [3 sentence writing limit]. Why is $r(6,7,8) \geq r(5,6,7)$?

3. [4 sentence writing limit] Give a combinatorial argument for the following identity, where r, m, and k are non-negative integers.

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$$

4. Consider a 9×9 checkerboard (see Figure 1 below) where on each square there is one pigeon. At the sound of a whistle, each pigeon moves one square to the left, right, up or down. (That is, the pigeon remains in the same row or column in which it started but is now one square away from its original position.) Let (a, b) denote the coordinates of a square, i.e. the square is in row a and column b. Prove that after the whistle sounds, there will be at least two pigeons on the same square. *Hint: how many squares have coordinates with even sum? how does the sum change when a pigeon moves?*

5. Find the coefficient of the term $x^{10}y^{10}$ in the polynomial

$$f(x, y) = x^5 y^5 (x - y)^5 (x + y)^5.$$

Hint: $(x - y)(x + y) = (x^2 - y^2)$. *Second hint: I have not made a typographical error. Third hint: no clarifying questions on this problem will be taken.*

6. Recall the secretary problem, where a secretary has to walk a number of blocks from home to work and, in doing that walk, he is only ever allowed to walk a block north or east until arriving at his destination. We are in the year 2024, which is equal to $\binom{21+3}{3}$. Prove that

$$\binom{21+3}{3} = \binom{23}{2} + \binom{22}{2} + \binom{21}{2} + \cdots + \binom{2}{2}.$$

One way to prove this identity is to count the number of distinct walks between the coordinates $(0, 0)$ and $(3, 21)$ in two different ways. (Consider walking from $(0, 0)$ to $(1, 0)$ as walking a block East, while walking from $(0, 0)$ to $(0, 1)$ as walking a block North.) Doing so would offer a combinatorial proof. Prove this identity in any manner that you choose.

7. The complete graph on n vertices is denoted K_n – the graph has n vertices and any two are connected by an edge.

   (a) How many edges does the graph have?

   (b) Consider placing pebbles on the vertices of this graph. Let us now label the vertices of the graph $v_1, v_2, \ldots, v_n$ such that vertex v_i has capacity to hold i pebbles. (So, vertex labelled v_1 has capacity 1, vertex labelled v_2 has capacity 2, etc.) How many

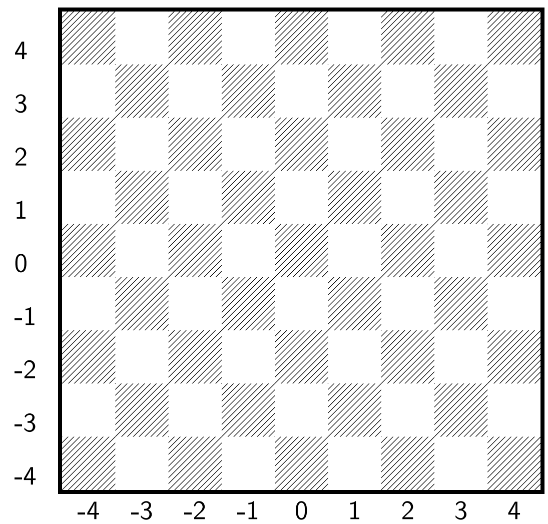

Figure 1: a pigeon is placed on each square

pebbles $p(n)$ must be placed before guaranteeing that one of the vertices exceeds its capacity?

(c) In the previous step, you determined $p(n)$. How many ways are there to place $p(n)$ pebbles on the vertices of this graph?

(d) Let us consider the specific case of $n = 3$. If a placement of $p(3)$ pebbles has been made (and so some vertex has exceeded capacity), then a *pebbling move* may be made, where a pebbling move consists of picking up $i + 1$ pebbles from vertex v_i, "throwing" i away and putting the other on a neighboring vertex. Which placements of $p(3)$ pebbles would allow for there to be a sequence of pebbling moves that throws away all but one pebble?