Combinatorics - MATH 0345

Exam 1

March 15, 2018

Name:
Honor Code Pledge:

Signature:

Directions: Please complete **six of the eight** problems. Justify all solutions — partial work receives partial credit. Each problem is worth ten points. Answers may be left in terms of factorials, binomial coefficients, products and sums of these, and the like. Some problems have a writing limit – the limit is indicated at the beginning of the problem. There is a time limit of 2 hours.

1. Give combinatorial reasoning to establish the following identity. (Please no algebraic manipulations before starting on it.)

$$\binom{n+m}{2} = \binom{n}{2} + \binom{m}{2} + mn$$

2. [Brualdi] A secretary works in a building located nine blocks east and eight blocks north of his home. Every day he walks 17 blocks to work. (A map is attached.) How many different routes are possible for him?

3. How many integers from the set $\{1, \ldots, 100\}$ must one pick to guarantee that from the subset chosen there are two that sum to 101? Can you also show that your bound is "sharp"? That is, what is a maximum size set such that no two add to 101?

4. [Brualdi] Prove that

$$r(\underbrace{3, 3, \ldots, 3}_{k+1}) \leq (k+1)(r(\underbrace{3, 3, \ldots, 3}_{k}) - 1) + 2.$$

5. Find the coefficient of the term $x^{4k}y^{4k}$ in the polynomial

$$f(x, y) = x^{2k}y^{2k}(x-y)^{2k}(x+y)^{2k}.$$

(This is a question derived from a paper that I wrote with Alec Cooper('13), Oleg Pikhurko and Greg Warrington entitled *Martin Gardner's minimum no-three-in-a-line problem.*)

6. Sperner's Theorem determined the maximum size of an antichain in the collection of all subsets of $[n] := \{1, 2, \ldots, n\}$. We now turn to an enumerative question: *how many* antichains are there? (I am not asking for you to count the maximum-size antichains, I'm asking for you to count all antichains.). Please do this when $n = 2$ and $n = 3$.

7. How many solutions to the following equation are there satisfying the following equation if $x_i \geq i$ for $1 \leq i \leq 100$?

$$x_1 + x_2 + \ldots + x_{100} = 25,000$$

8. Prove the following identity.

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{m} = \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} 2^k$$