Combinatorics - MATH 0345

Exam 1

March 15, 2018

Name:
Honor Code Pledge:

Signature:

Directions: Please complete **seven of the eight** problems. Justify all solutions — partial work receives partial credit. Each problem is worth ten points. Answers may be left in terms of factorials, binomial coefficients, products and sums of these, and the like. Some problems have a writing limit – the limit is indicated at the beginning of the problem. There is a time limit of 2 hours.

1. [**Writing limit: 4 sentences**] A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

   A common error on this problem was to assume that the intersection of all sets was non-empty. That is, people assumed that there exists an element x common to all sets in the collection - this is not true.

2. [**Writing limit: 4 sentences**] Consider the faces of a polyhedron. We proved that there are two faces bounded by the same number of edges. In fact, there are always at least *two different pairs* of faces with the same number of edges. Prove this.

3. [**Writing limit: 4 sentences**] Give combinatorial reasoning to establish the following identity.

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

4. Use the pigeonhole principle to show that any set S consisting of 102 numbers from the set $\{1, 2, \ldots, 200\}$ must have two elements x and y, with $x \neq y$, such that there exists a $z \in S$ and $x + y = z$.

The Twelvefold Way

distinct or identical		number of candies		
candies	bags	any	≤ 1	≥ 1
D	D			
I	D			
D	I			
I	I			

Table 1: Good stuff

5. [**Writing limit: 3 sentences**] In class we proved that the Ramsey number $r(4,4) \leq 18$. How can one establish that $r(4,4) \geq 18$? (You needn't establish this, you just need to tell me how to do this.) [**Writing limit: 5 sentences**] If one wanted to establish the upper bound for $r(4,4) \leq 18$ with a brute-force approach (i.e. not using the pigeonhole principle), then how many cases are there to check?

6. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

satisfy $x_1 \geq 2, x_2 \geq 0, x_3 \geq -5$, and $x_4 \geq 8$? [Brualdi]

7. Give the coefficient of $x_1^3 x_2^3 x_3 x_4^2$ in the expansion of

$$(x_1 - x_2 + 2x_3 - 2x_4)^9.$$

Next give the coefficient of $x_1^2 x_2^3 x_3^3 x_4^2$.

8. Consider the table of the Twelvefold Way above. Let's consider the case when the number of candies is 3 and the number of bags is 3. Some of the entries on this table are then trivial. Which ones are these and what is the value for each, respectively?

(Note: this table was typeset using the following on-line super-cool resource, `https://www.tablesgenerator.com/`.)