Combinatorics - MATH 0345

Exam 1

October 14, 2021

Name:
Honor Code Pledge:

Signature:

Directions: Please complete **six of the seven** problems. Justify all solutions — partial work receives partial credit. Each problem is worth ten points. Answers may be left in terms of factorials, binomial coefficients, products and sums of these, and the like. Some problems have a writing limit – the limit is indicated at the beginning of the problem. There is a time limit of 2 hours.

1. Consider the multiset $\{10 \cdot a, 1, 2, 3, \ldots, 10\}$ of size 20. Determine the number of its 10-combinations.

2. Count the number of integers between 2,000 and 3,000 (inclusive) that have the property that the sum of their digits is 10. For example, the integer 2071 has this property since $2 + 0 + 7 + 1 = 10$.

3. [**4 sentences limit**] If five points are chosen at random with a square of side length 1, what is the smallest distance that one can guarantee to exist for at least one pair of points in the collection? Why? *Hint: PHP*

4. *Ramsey Theory* Give a red-blue coloring of the edges of K_4 so that there is neither a red path on four vertices nor a blue path on four vertices (see the figure below for what a path on four vertices looks like). Next prove that regardless of how one red-blue colors the edges of K_5, there exists a monochromatic copy of this graph.

5. [**4 sentences limit**] Give a combinatorial proof of the following identity. Begin the proof as such: "There are n people and the left side counts..."

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = 2^{n-1}.$$

6. Use induction on m to establish that

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k}$$

equals $(-1)^m \binom{n-1}{m}$. If you take advantage of a well-known identity, which one is it?

7. Prove, that for every integer $n > 1$,

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \cdots + (-1)^{n-1} n \binom{n}{n} = 0.$$

Hint: Differentiate the binomial formula for $(1+x)^n$ and then replace x by -1.