Name ____
ID number____________________________________
Sections **B**

Calculus II (Math 122) Final Exam, 17 May 2013

This is a closed book exam. No notes or calculators are allowed. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, $\ln(), \exp()$, fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the honor code (I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.) in the space provided below. Good luck!

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Honor Code:

Signature:

1. [5 points] Solve the following first-order linear differential equation.

$$y' + y = \sin(e^x)$$

2. [1 point each] **Fill in the blank** Please complete the proof by justifying why each step is true.

   Claim: If population growth follows the logistic model, then population growth begins to slow when the population reaches half of the carrying capacity.

   PROOF: Let us consider the logistic differential equation $\frac{dP}{dt} = rP(M-P)$ for some ____________________ r and M.

   As we wish to determine when the rate of growth begins to slow, we consider the _________ derivative. Thus, we will find $\frac{d^2P}{dt^2}$.

   So, we compute,
   $\frac{d^2P}{dt^2} = rP(-\frac{dP}{dt}) + (M-P)r\frac{dP}{dt}$ by the ___________________ for differentiation.

   By a _______________, this equals $rP(-rP(M-P)) + (M-P)r^2P(M-P)$. We may distribute to obtain

   ____,

   collect like terms to obtain

   ________________________________,

   and factor, to obtain

   $r^2P(2P-M)(P-M)$.

   This quantity is zero when $P = 0$, $P = M$ or ______________. We now restrict P to the _______ interval $(0, M)$. So we see that concavity changes when $P = \frac{M}{2}$. That is, the first derivative changes from __________ to __________ at this population level. $\square$

3. [5 points each] Determine whether the following series converge or diverge. You must state or clearly demonstrate what test you are using to determine convergence and justify its use. Heuristic or intuitive reasoning will not get full credit, though it may help you get started.

   (a) $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+5}$

   (b) $\sum_{n=1}^{\infty} \frac{1}{2+\sin n}$

4. [15 points] Approximate $f(x) = x \ln x$ by a Taylor polynomial of degree $n = 3$ at the number $a = 1$. Then use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the interval $0.5 \leq x \leq 1.5$.

5. [5 points each] Evaluate each of the following integrals. State the domain of your answer.

(a) $\int \sqrt{1+x^2}dx$

(b) $\int x \arctan(x) dx$

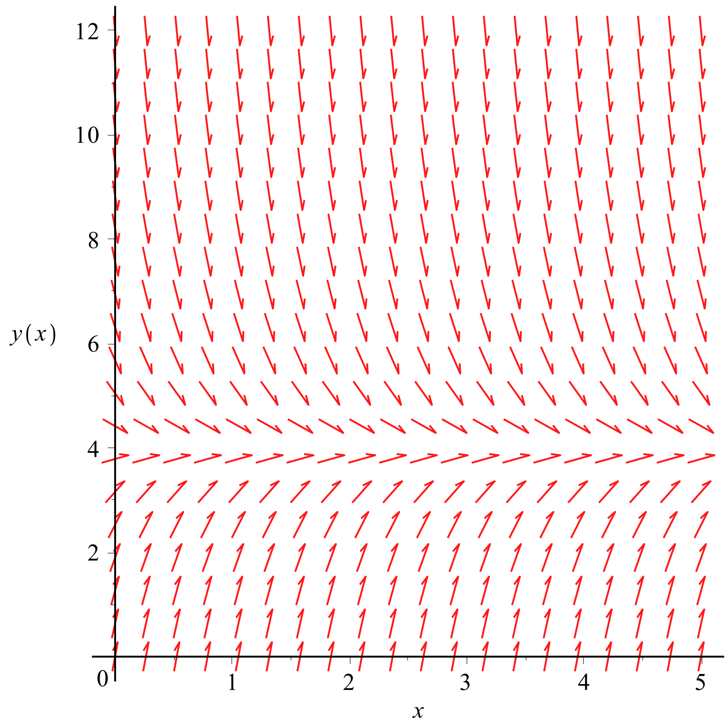

Figure 1: The direction field

6. [15 points] A portion of the direction field for the differential equation $\frac{dy}{dx} = 12 - 3y$ is shown.

   (a) Complete the sketch of the direction field for the portion that has been "covered over". That is, sketch the direction field for $0 \leq x \leq 5$ and $4 \leq y \leq 12$.

   (b) Sketch the graph of the solution that satisfies the given initial condition $(4, 2)$.

   (c) Use Euler's method with step size of $h = 0.2$ to estimate $y(4.6)$ (using the same initial conditions as above).

7. [10 points] The differential equation in the previous problem is a separable differential equation, so it can be solved analytically. Solve the differential equation subject to the initial condition given in the previous problem.

8. [10 points] Now solve the same differential equation by finding the Taylor series for y in powers of $(x-1)$.

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$