Name ____
ID number______________________________________
Sections **B**

Calculus II (Math 122) Final Exam, 11 December 2013

This is a closed book exam. Notes and calculators are not allowed. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, $\ln(), \exp()$, fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the Honor Code (I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.) in the space provided below. Good luck!

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

Honor Code:

Signature:

1. [10 points] Below is a sketch of the direction field for a differential equation. Sketch the graphs of the solutions that satisfy the given initial conditions $y(4) = 1$ and $y(-6) = 3$.

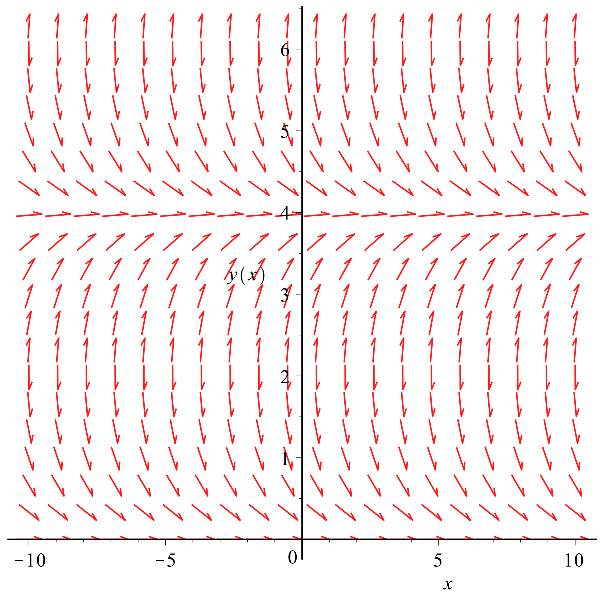

Figure 1: The direction field for $y' = -\tan(\frac{\pi y}{4})$

Now find **all** the equilibrium solutions (not just those seen in the sketch).

2. [5 points] Consider the logistic differential equation $\frac{dP}{dt} = rP(M - P)$ and recall that the second derivative of P is given by $r^2P(2P - M)(P - M)$. Assume that as you collect data on a population P over time t, you notice that the data fits the logistic model and you notice that the rate of change of the population slows when the population hits $5,000$. What might you predict about the carrying capacity M of the population your are observing. Justify your answer.

3. [5 points each] Determine whether the following series converges or diverges. You must state or clearly demonstrate what test you are using to determine convergence and justify its use. Heuristic or intuitive reasoning will not get full credit, though it may help you get started.

   (a) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

   (b) $\sum_{n=1}^{\infty} \frac{n \sin^2(n)}{1+n^3}$

4. [10 points] Solve the following separable differential equation.

$$\frac{dy}{dx} = \frac{x(y^2+1)}{x+1}$$

5. [10 points] Evaluate $\int xe^{-x^3}\,dx$ as an infinite series.

Give an approximation of $\int_0^1 xe^{-x^3}\,dx$ **and** bound the error of your approximation.

6. [10 points] A tank contains 100 gallons of fresh water. A solution containing 1 lb./gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find an expression for the amount of fertilizer F in the tank at time t.

7. [10 points] Demonstrate 3 iterations in Euler's method with step size of $h = 0.5$ for the differential equation and initial condition given.

$$y' = 1 + y, \qquad y(0) = 1$$

8. [10 points] Solve the following differential equation: $4x^3y + x^4y' = \sin^3(x)$.

9. [10 points] Find an equation of the tangent to the curve given by $x = \sin^3(\theta),\ y = \cos^3(\theta)$ at $\theta = \pi/6$.

10. [10 points] **Length is independent of parametrization!** The following exercise is meant to demonstrate that the length of a curve is independent of the parametrization given to it (so long as we don't "double-back"). Below are two different parameterizations of the unit semi-circle. Show that each provides arc length of π.

$$x = \cos(2t), \quad y = \sin(2t), \quad 0 \leq t \leq \pi/2$$

$$x = \sin(\pi t), \quad y = \cos(\pi t), \quad -1/2 \leq t \leq 1/2$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$