Name ________________________________

ID number________________________________

Please circle your section: A: 8:00 - 8:50, B: 9:05 - 9:55, C 10:10 - 11 or D 11:15 - 12:05

Calculus II (Math 122) Final Exam, 11 December 2007

This is a closed book exam. No notes or calculators are allowed. To receive credit you must show your work. When you are finished please write and sign the honor code, (I have neither given nor received unauthorized aid on this exam) in the space provided below.

1	10
2	8
3	24
4	8
5	12
6	16
7	16
8	20
Total	

Honor Code:

Signature:

1. (2 points each) Mark each of the following true (T) or false (F).

(a) F $\int_1^{3+\sqrt{2}} \frac{1}{t}\,dt = \ln(3) + \ln\sqrt{2}$.   Integral equals $\ln(3+\sqrt{2}) \neq \ln(3) + \ln\sqrt{2}$.

(b) T If for the sequence $\{a_k\}$ we have $a_k > a_{k+1} > 0$ for all integers $k > 0$ then $\lim_{k\to\infty} a_k$ exists. This is what the Monotonic Sequence Theorem says.

(c) F If $a_k > a_{k+1} > 0$ for all integers $k > 0$ then the series $\sum_{k=1}^{\infty} a_k$ converges.
It is necessary for the terms to go to zero, but not sufficient.

(d) T When the series $\sum_{k=0}^{\infty} (10x)^k$ converges, its sum is $\frac{1}{1-10x}$.
This is a geometric series with common ratio $10x$.

(e) F If the series $\sum_{k=0}^{\infty} a_k$ diverges and the series $\sum_{k=0}^{\infty} b_k$ diverges, then the series $\sum_{k=0}^{\infty} (a_k - b_k)$ must diverge.
As a counterexample, consider $\sum_{k=0}^{\infty} \frac{1}{k}$ and $\sum_{k=0}^{\infty} \frac{1}{k}$.

2. (8 points) Determine the convergence set (the interval of convergence) for $\sum_{k=1}^{\infty} \frac{(2x-3)^k}{k+16}$.

We consider using the Ratio test:

$$\lim_{k\to\infty} \left| \frac{(2x-3)^{k+1}}{k+17} \cdot \frac{k+16}{(2x-3)^k} \right| = |2x-3| \cdot \lim_{k\to\infty} \left| \frac{k+16}{k+17} \right| = |2x-3|$$

The Ratio test implies absolute convergence when $|2x-3| < 1 \iff$
$-1 < 2x-3 < 1 \iff 2 < 2x < 4 \iff 1 < x < 2$.
It also implies divergence for $x > 2$ and $x < 1$, but is inconclusive if $x = 1$ or $x = 2$ - so we test these points separately.

When $x = 1$ the series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k+16}$, which converges by the alternating series test. (Note that it is the tail of the alternating harmonic series.)

When $x = 2$ the series is $\sum_{k=1}^{\infty} \frac{1}{k+16}$, which diverges - it is the tail of the harmonic series.

Thus the interval of convergence is $[1, 2)$.

3. (8 points each) Determine if the following series converge or diverge. For alternating series that converge determine if the convergence is absolute or conditional. You must state or clearly demonstrate what test you are using to determine convergence. You will not receive full credit for just heuristic reasoning.

(a) $\sum_{k=1}^{\infty} ke^{-k^2}$.

We will use the integral test and so consider $\int_1^{\infty} k e^{-k^2} dk$, where k is now a cont. variable.

We obtain $\lim_{c \to \infty} -\frac{1}{2}\int_1^c -2k e^{-k^2} dk = -\frac{1}{2} \lim_{c\to\infty} e^{-k^2}\Big|_1^c$

$= -\frac{1}{2} \lim_{c\to\infty} e^{-c^2} - e^{-1} = \frac{1}{2e}$

$\Rightarrow$ Thus by the integral test the series converges.

(b) $\sum_{k=4}^{\infty} \frac{2-k^2}{(k-3)^2}$.

We apply the nth term test for divergence.

Since $\lim_{k\to\infty} \frac{2-k^2}{(k-3)^2} = \lim_{k\to\infty} \frac{2-k^2}{k^2-6k+9} \overset{L.H}{=} \lim_{k\to\infty} \frac{-2k}{2k-6} = \lim_{k\to\infty} \frac{-2}{2} = -1$

and $-1 \neq 0$ we have that this series diverges.

(c) $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k^2+3k}{k^3+4}$.

We apply the A.S.T., noting that the series does alternate, that

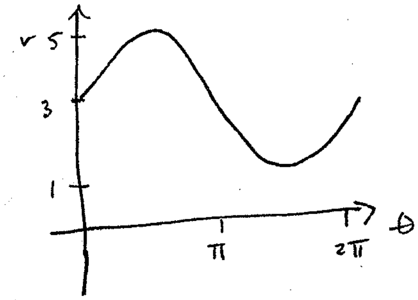
$\lim_{k\to\infty} \frac{k^2+3k}{k^3+4} = \lim_{k\to\infty} \frac{2k+3}{3k^2} = \lim_{k\to\infty} \frac{2}{6k} = 0$ and the terms decrease since

$\frac{d\left(\frac{x^2+3x}{x^3+4}\right)}{dx}$ is negative for x large enough. Which implies the series converges.

It does not converge absolutely by comparing $\sum_{k=1}^{\infty} \frac{k^2+3k}{k^3+4}$ to $\sum_{k=1}^{\infty} \frac{1}{k}$, a divergent series. Note that $\lim_{k\to\infty} \frac{\frac{k^2+3k}{k^3+4}}{\frac{1}{k}} = 1$.

4. (8 points) Find the area inside $r = 3 + 2\sin(\theta)$ and outside $r = 4$.

First we sketch $r = 3 + 2\sin\theta$ on a Cartesian system and then transfer the sketch to polar coordinate system. (See next page.)



We must first find the points of intersection.

$$4 = 3 + 2\sin\theta$$

$$1 = 2\sin\theta$$

$$\frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

So the area is

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(3+2\sin\theta)^2 \, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4)^2 \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 9 + 12\sin\theta + 4\sin^2\theta \, d\theta - 8\int_{\pi/6}^{5\pi/6} d\theta$$

$$= \int_{\pi/6}^{5\pi/6} -\frac{7}{2} + 6\sin\theta + 2\sin^2\theta \, d\theta = \int_{\pi/6}^{5\pi/6} -\frac{7}{2} + 6\sin\theta + 1 - \cos 2\theta \, d\theta$$

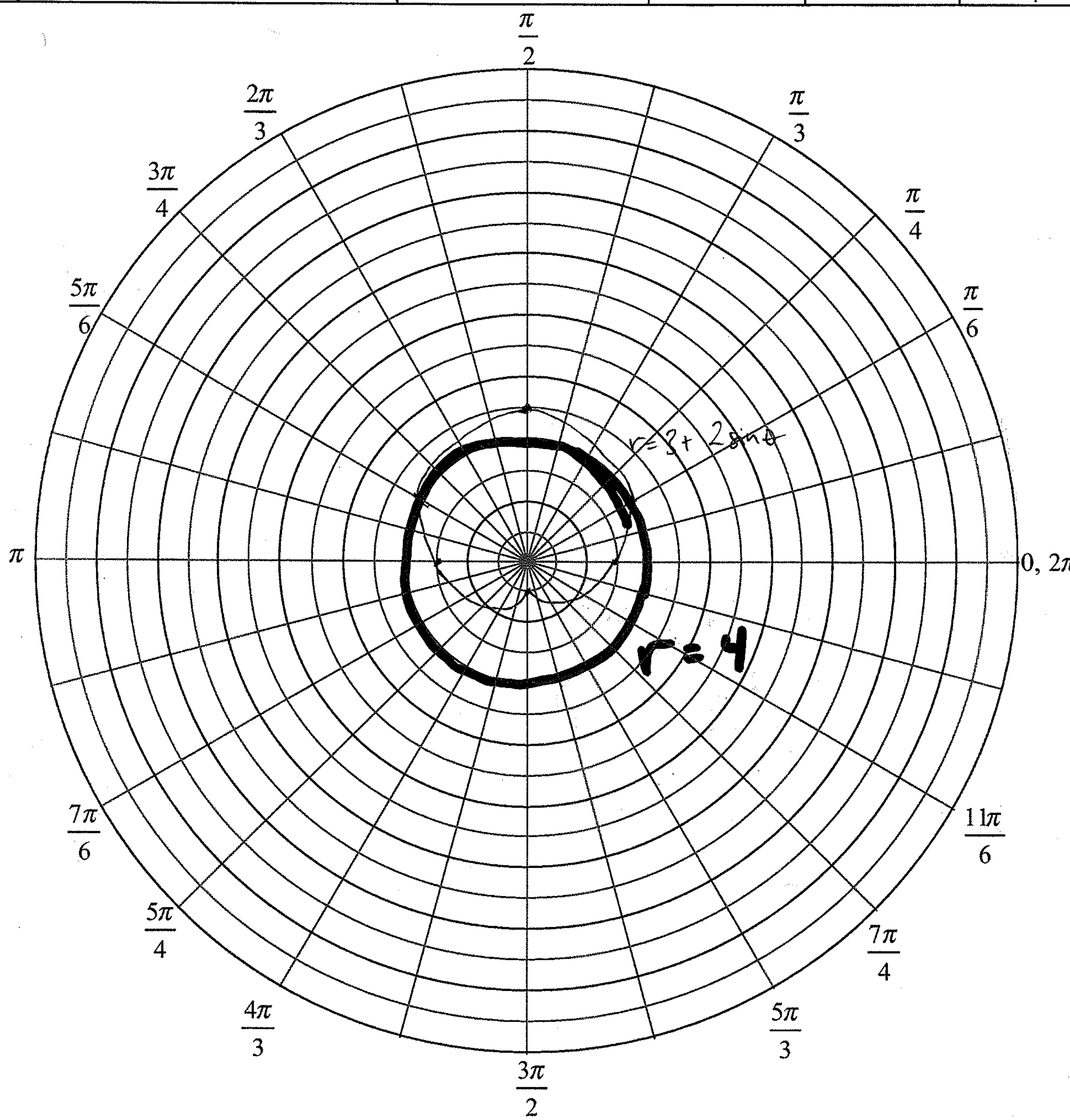
$$= \int_{\pi/6}^{5\pi/6} -\frac{5}{2} + 6\sin\theta - \cos 2\theta \, d\theta = -\frac{5}{2}\theta - 6\cos\theta - \frac{1}{2}\sin 2\theta \Big|_{\pi/6}^{5\pi/6}$$

$$= \left(-\frac{5}{2}\cdot\frac{5\pi}{6} - 6\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)\right) - \left(-\frac{5}{2}\frac{\pi}{6} - 6\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{-20\pi}{12} + 6\sqrt{3} + \frac{1}{2}\sqrt{3}$$

$$= \frac{13\sqrt{3}}{2} - \frac{5\pi}{3}$$

Name: Description:	Section/Page Numbers:	Problems:	Date:	Page 1



5. (12 points) The folium of Descartes is defined by the parametric equations

$$x = \frac{3t}{1+t^3}, \quad y = \frac{3t^2}{1+t^3}.$$

(a) Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{(1+t^3)6t - (3t^2)(3t^2)}{(1+t^3)^2}}{\frac{(1+t^3)3 - (3t)(3t^2)}{(1+t^3)^2}} = \frac{3t\left[2(1+t^3) - 3t^3\right]}{3(1+t^3) - 9t^3}$$

$$= \frac{6t + 6t^4 - 9t^4}{3 - 8t^3}$$

$$= \frac{6t - 3t^4}{3 - 8t^3} = \frac{3t(2-t^3)}{3-8t^3}$$

$$= \frac{t(2-t^3)}{1-2t^3}$$

(b) Find the points on the curve where the tangent lines are horizontal.

The tangent lines are horizontal when $\frac{dy}{dt} = 0$ which occurs when

$$(1+t^3)6t - (3t^2)(3t^2) = 0$$

$$6t + 6t^4 - 9t^4 = 0$$

$$6t - 3t^4 = 0$$

$$3t(2 - t^3) = 0 \quad \Rightarrow \quad t = 0 \text{ or } t = \sqrt[3]{2}$$

$\Rightarrow$ So we obtain points $x(0) = \frac{0}{1}$, $y(0) = \frac{0}{1}$ $\Rightarrow (0,0)$

and $x(\sqrt[3]{2}) = \frac{3\sqrt[3]{2}}{3} = \sqrt[3]{2}$, $y(\sqrt[3]{2}) = \frac{3(\sqrt[3]{2})^2}{3} = (\sqrt[3]{2})^2 \Rightarrow (\sqrt[3]{2}, (\sqrt[3]{2})^2)$

6. (8 points each) Determine the following.

(a) $\int \sqrt{9 - x^2}\, dx.$

$$= \int \sqrt{9 - 9\sin^2\theta}\; 3\cos\theta\, d\theta$$

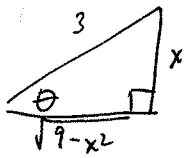
Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

restrict $0 < \theta \leq \pi$

$$= \int \sqrt{9(1-\sin^2\theta)}\; 3\cos\theta\, d\theta$$

$$= \int 3\cos\theta\; 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta$$



$$= 9\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 9\left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right] + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + 9 \cdot \frac{2}{4}\,\frac{x}{3}\,\frac{\sqrt{9-x^2}}{3} + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9x}{18}\sqrt{9-x^2} + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{2}\sqrt{9-x^2} + C$$

(b) $\int_0^4 \frac{1}{(x-3)^2}\, dx.$

Let $u = x - 3$ $\quad du = 1\,dx$

Then this integral is $\int_{-3}^{1} \frac{1}{u^2}\, du$, but it is an improper integral b/c of discontinuity at $u = 0$

So,

$$\lim_{b\to 0^+} \int_b^1 \frac{1}{u^2}\, du + \lim_{c\to 0^-} \int_{-3}^{c} \frac{1}{u^2}\, du = \lim_{b\to 0^+} \left.\frac{-1}{u}\right|_b^1 + \lim_{c\to 0^-} \left.-\frac{1}{u}\right|_{-3}^{c}$$

$$= \lim_{b\to 0^+}\left(\frac{-1}{1} + \frac{1}{b}\right) + \lim_{c\to 0^-}\left(-\frac{1}{c} + \frac{1}{3}\right) = \infty$$

(c) $\lim_{x\to\infty}\left(1 + \frac{2}{x}\right)^{5x}.$

Let $y = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{5x}$ then $\ln y = \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^{5x}$

$$= \lim_{x\to\infty} 5x \ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{5x}}$$

$$\overset{L.H}{=} \lim_{x\to\infty} \frac{\frac{x}{x+2}\cdot\frac{-2}{x^2}}{\frac{-1}{5x^2}} = \lim_{x\to\infty} 10\,\frac{x}{x+2}$$

$$= 10$$

$$\Rightarrow \ln y = 10 \Rightarrow y = e^{10}$$

7. (8 points each) Solve the differential equations and initial value problems

(a) $\frac{dy}{dx} = y(y-5)$; $y(0) = 1.$

This is a separable differential equation so we write

$$\int \frac{dy}{y(y-5)} = \int dx .$$

To integrate the LHS we must use partial fraction decomposition:

$$\frac{1}{y(y-5)} = \frac{A}{y} + \frac{B}{y-5} \Rightarrow A(y-5) + By = 1 .$$ Setting $y=0$ we find $A = -1/5$, $y=5$ we

And $B = \frac{1}{5}$. So our integration problem becomes.

$$\int \frac{-1/5}{y} dy + \int \frac{1/5}{y-5} dy = \int dx \Leftrightarrow -\frac{1}{5} \ln|y| + \frac{1}{5} \ln|y-5| = x + C \Rightarrow \frac{1}{5} \ln\left|\frac{y-5}{y}\right| = x + C$$

$$\Rightarrow \ln\left|\frac{y-5}{y}\right| = 5x + C \Rightarrow \frac{y-5}{y} = e^{5x+C} \Rightarrow 1 - \frac{5}{y} = e^{5x+C} \Rightarrow y = \frac{5}{1-Ce^{5x}}$$

We now use the initial condition to solve for C.

$$1 = \frac{5}{1-C} \Rightarrow C = 4 \Rightarrow y = \frac{5}{1-4e^{5x}}$$

(b) $y' = y + 2xe^{2x}$; $y(0) = 3.$

Rewriting as $y' - y = 2xe^{2x}$ we note that this is a linear diff. eq. with $P(x) = -1$. Thus our integrating factor is $I(x) = e^{\int -1 dx} = e^{-x}$.

Thus, by multiplying by $I(x)$, we obtain:

$$y'e^{-x} - e^{-x}y = 2xe^{x}$$

$$\frac{d(ye^{-x})}{dx} = 2xe^{x} \Rightarrow \text{We now integrate } \int \frac{d(ye^{-x})}{dx} dx = \int 2xe^{x} dx$$

To integrate the RHS we use int. by parts with $u = x$, $dv = e^x dx$, $du = 1 dx$, $v = e^x$

So we obtain

$$ye^{-x} = 2\left[xe^{x} - \int e^{x} dx\right]$$

$$ye^{-x} = 2xe^{x} - 2e^{x} + C \Rightarrow y = 2xe^{2x} - 2e^{2x} + Ce^{x}$$

We solve for C: $3 = 0 - 2 + C \Rightarrow C = 5 \Rightarrow y = 2xe^{2x} - 2e^{2x} + 5e^{x}$

8. A tank with a capacity of 500 gallons originally contains 200 gallons of water in which is dissolved 100 pounds of salt. Brine containing 1 pound of salt per gallon enters the tank at at rate of 3 gal/min. The well stirred mixture is allowed to flow out at 2 gal/min.

(a) (8 points) Find the equation that will enable you to compute the amount of salt in the tank at any time prior to the instant the tank is about to overflow.

Let $y(t)$ denote the amount of salt in the tank at time t.

We first find $\frac{dy}{dt}$ = Rate in − Rate out.

Rate in: $\frac{1 \text{ lb}}{\text{gal}} \cdot \frac{3 \text{ gal}}{\text{min}} = \frac{3 \text{ lb}}{\text{min}}$

Rate out: $\frac{y \text{ lb}}{200+t \text{ gal}} \cdot \frac{2 \text{ gal}}{\text{min}} = \frac{2y}{200+t}$ lb/min since concentration $= \frac{y(t)}{V(t)}$ and $V(t) = 200+3t-2t = 200+t$

Thus $\frac{dy}{dt} = 3 - \frac{2y}{200+t}$, which can be written as $\frac{dy}{dt} + \frac{2}{200+t} y = 3$

This is a 1st order linear diff. eq. with integrating factor $I(t) = e^{\int \frac{2}{200+t} dt} = e^{2\ln|200+t|} = e^{\ln(200+t)^2} = (200+t)^2$. We multiply through by $I(t)$ to obtain

$(200+t)^2 \frac{dy}{dt} + 2(200+t)y = 3(200+t)^2 \Rightarrow \frac{d((200+t)^2 \cdot y)}{dt} = 3(200+t)^2$

Integrating both sides we obtain $(200+t)^2 \cdot y = \int 3(200+t)^2 dt$

$= \frac{3(200+t)^3}{3} + C_1$

$\Rightarrow y = (200+t) + \frac{C_1}{(200+t)^2}$. We use initial condition $y(0) = 100$ to solve for C_1. I omit the step.

(b) (6 points) Find the concentration (in pounds per gallon) of salt in the tank when the tank is totally full (that is, at the instant it is about to overflow).

Concentration is: $c(t) = \frac{y(t)}{V(t)} = \frac{(200+t) + \frac{C_1}{(200+t)^2}}{(200+t)}$

$= 1 + \frac{C_1}{(200+t)^3}$

So need to find $c(300)$ since at $t = 300$ $V = 500$.

$C(300) = 1 + \frac{C_1}{(500)^3}$

(c) (6 points) Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

So we consider

$$\lim_{t \to \infty} c(t) = \lim_{t \to \infty} 1 + \frac{c_1}{(200+t)^3}$$

$$= 1.$$