Calculus II - Exam 3 - Fall 2007

November 15, 200~~6~~ 7

Name:
Honor Code Statement:

Directions: Justify all answers/solutions. Calculators are not permitted.

(8) 1. Test the series for convergence or divergence.

$$\Sigma_{n=1}^{\infty}(-1)^{n-1}\frac{n}{e^n}$$

We'll apply the Alternating Series Test, since the series is of the form $\sum_1^\infty (-1)^{n-1} a_n$ where $a_n > 0$ for all n.

First note that $\lim_{n\to\infty} \frac{n}{e^n} \overset{L.H.}{=} \lim_{n\to\infty} \frac{1}{e^n} = 0$.

And note that $a_{n+1} \leq a_n$, since $\frac{d\left(\frac{x}{e^x}\right)}{dx} = \frac{e^x \cdot 1 - x \cdot e^x}{e^{2x}}$

$$= \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x}$$

is negative for $x > 1$.

Thus the A.S.T. says that this series converges.

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\Sigma_{n=1}^{\infty} \frac{(-1)^n \arctan(n)}{n^2}$$

We consider the absolute series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \arctan(n)}{n^2} \right|$ which equals $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2}$ since $\arctan(n) > 0$ for $n \geq 1$..

Recall that $\arctan(n) \leq \frac{\pi}{2}$ for all n, thus

$$\frac{\arctan(n)}{n^2} \leq \frac{\pi/2}{n^2} \quad \text{for all } n.$$

Note that $\sum_{n=1}^{\infty} \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent since $\sum_1^{\infty} \frac{1}{n^2}$ is a convergent p-series.

Thus by the comparison test the above absolute series is convergent and so the given series is absolutely convergent.

3. Find the radius of convergence and interval of convergence for each of the given power series.

(a)

$$\Sigma_{n=1}^{\infty}(-1)^n\frac{n^2x^n}{2^n}$$

To determine the interval of convergence we'll apply the ratio test. So we compute

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2x^{n+1}}{2^{n+1}}}{\frac{n^2x^n}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\frac{1}{2}x\right| = \frac{|x|}{2}\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^2\right|$$

$$= \frac{|x|}{2}$$

When this limit is less than 1 the series converges absolutely. Thus when $\frac{|x|}{2} < 1$ or $|x| < 2$ the series converges.

We now consider the endpoints of this interval.

When $x=2$ the series is $\sum_1^{\infty}(-1)^n n^2$, which diverges by the n^{th} term test for divergence.

When $x=-2$ the series is $\sum_1^{\infty} n^2$, which diverges by the n^{th} term test for divergence.

Thus the interval of convergence is $(-2, 2)$ and the radius of convergence is 2.

(b)

$$\Sigma_{n=1}^{\infty} n!(2x-1)^n$$

We'll apply the absolute ratio test. So we compute

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n} \right| = \lim_{n\to\infty} \left|(n+1)\,(2x-1)\right|$$

$$= |2x-1| \cdot \lim_{n\to\infty} |n+1| .$$

This limit is ∞ unless $2x-1=0$, in which case $x=\frac{1}{2}$.

Thus the series converges only when $x=\frac{1}{2}$, and the radius of convergence is zero.

4. Find a power series representation for the given function and determine the interval of convergence.

$$f(x) = \frac{1}{x+10}$$

Let's recall that a geometric series of the form $\sum_{n=1}^{\infty} a r^{n-1}$ converges to $\frac{a}{1-r}$ when $|r|<1$.

We can rewrite

$$\frac{1}{x+10} = \frac{1}{10-(-x)} = \frac{1}{10\left(1-\left(\frac{-x}{10}\right)\right)} = \frac{1/10}{1-\left(\frac{-x}{10}\right)}$$

Thus we may think of $a = \frac{1}{10}$ and $r = \frac{-x}{10}$

So $f(x)$ has a power series representation $\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)\left(\frac{-x}{10}\right)^{n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-1}}{10^{n}}$,

which converges when $\left|\frac{-x}{10}\right| < 1 \Leftrightarrow |x| < 10$, that is on the interval $(-10, 10)$.

It does not converge at $x = -10$ by the nth term test for divergence.

or at $x = 10$ for the same reason.

5. Use a power series to approximate the definite integral. Give an error estimate for your answer. (As calculators are not permitted, you need not compute/simplify the approximation and error estimate.)

$$\int_0^{0.5} \frac{1}{1+x^5}dx$$

First we note that we may write

$\frac{1}{1+x^5}$ as $\frac{1}{1-(-x^5)}$ which may be thought of

as the sum of a geometric series, This has a power

series representation $\sum_{n=1}^{\infty}(-x^5)^{n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}x^{5n-5}$ which

converges when $|-x^5|<1$, that is when $|x|<1$

Thus we may write

$$\int_0^{0.5}\frac{1}{1+x^5}dx = \int_0^{0.5}\sum_{n=1}^{\infty}(-1)^{n-1}x^{5n-5}dx$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{5n-4}}{5n-4}\Bigg|_0^{0.5}$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{1}{2}\right)^{5n-4}}{5n-4}$$

This is an alternating (convergent) series. We can use the first, say, three non-zero terms to approximate the sum and note that our error is at most the 4th non-zero term.

Sum is approximately $\left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^6}{6} + \frac{\left(\frac{1}{2}\right)^{11}}{11}$ and the

error is at most $\left(\frac{1}{2}\right)^{16}/16$.

6. Represent $f(x) = \cos(x)$ as the sum of its Taylor series centered at $\frac{\pi}{2}$.

We will express cos x as a power series in the form $\sum_{n=0}^{\infty} c_n (x - \frac{\pi}{2})^n$. To do this we must find the coefficients, c_n. To do this we compute the derivatives of cos(x) and evaluate at $a = \frac{\pi}{2}$

$f(x) = \cos(x)$ $\quad f(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0 \quad \Rightarrow c_0 = \frac{0}{0!} = 0$

$f'(x) = -\sin(x)$ $\quad f'(\frac{\pi}{2}) = -\sin(\frac{\pi}{2}) = -1 \quad \Rightarrow c_1 = \frac{-1}{1!} = \frac{-1}{1!}$

$f''(x) = -\cos(x)$ $\quad f''(\frac{\pi}{2}) = -\cos(\frac{\pi}{2}) = 0 \quad \Rightarrow c_2 = \frac{0}{2!} = 0$

$f'''(x) = \sin(x)$ $\quad f'''(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1 \quad \Rightarrow c_3 = \frac{1}{3!}$

After which the derivatives "cycle", that is $f^{4k}(x) = f(x)$, $f^{4k+1}(x) = f'(x)$, $f^{4k+2}(x) = f''(x)$, $f^{4k+3}(x) = f'''(x)$.

Thus we see that only the odd coefficients are non-zero and

$c_{2n+1} = \frac{(-1)^{n+1}}{(2n+1)!}$

Thus the power series is, that is the Taylor series centered at $\frac{\pi}{2}$ is,

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$$

7. Find the length of the curve, where $x = \frac{y^4}{8} + \frac{1}{4y^2}$, $1 \leq y \leq 2$.

To find the length of this curve we will use the arc length formula.

$$L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

So first we find $1 + \left(\frac{dx}{dy}\right)^2$. First $\frac{dx}{dy} = \frac{y^3}{2} - \frac{1}{2y^3}$

and so $\left(\frac{dx}{dy}\right)^2 = \frac{y^6}{4} + \frac{1}{4y^6} - \frac{1}{2}$ and so $\left(\frac{dx}{dy}\right)^2 + 1 = \frac{y^6}{4} + \frac{1}{4y^6} + \frac{1}{2}$

$= \left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2$

Thus $L = \int_1^2 \sqrt{\left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2}\, dy$

$= \int_1^2 \frac{y^3}{2} + \frac{1}{2y^3}\, dy$

$= \frac{y^4}{8} - \frac{1}{4y^2}\Big|_1^2$

$= \left(2 - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right)$

$= \frac{32}{16} - \frac{1}{16} - \frac{2}{16} + \frac{4}{16}$

$= \frac{33}{16}$