Calculus II - Exam 3 - Fall 2006

November 16, 2006

Name:
Honor Code Statement:

Directions: Justify all answers/solutions. Calculators are not permitted.

1. Define what it means for a sequence to **diverge to infinity.**

The sequence $\{a_n\}$ diverges to infinity if for every number M there is an integer N s.t. for all ~~n~~ n larger than N, $a_n > M$.

2. Define what it means for M to be a **least upper bound** for a sequence $\{a_n\}$.

A number M is said to be a least upper bound if M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$.

Indicate which test you use

3. Indicate which series converge and which diverge. **JUSTIFY** your answer. You may **OMIT ONE** item from this question - indicate which you omit.

- $$\Sigma_{n=0}^{\infty}\frac{n!}{1,000,000^n}$$

Diverges by the n^{th} term test for divergence.

We have that $\lim_{n\to\infty} \frac{n!}{1{,}000{,}000^n} = \infty$

(Applying the Ratio Test is also valid.)

- $$\Sigma_{n=1}^{\infty}\frac{1}{3^{n-1}+1}$$

. (Give an upper bound for the sum.)

We have that $\frac{1}{3^{n-1}+1} \leq \frac{1}{3^{n-1}}$. Thus we have $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1} \leq \sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$

The latter is a geometric series w/ first term 1 and common ratio equal to $\frac{1}{3}$. It thus converges to $\frac{1}{1-1/3} = \frac{3}{2}$.

So by the Direct Comparison test $\sum \frac{1}{3^{n-1}+1}$ converges, and is bounded by $\frac{3}{2}$.

- $$\Sigma_{n=2}^{\infty}\frac{(\ln(n))^3}{n^3}$$

(Hint: compare to $\Sigma_{n=2}^{\infty}\frac{1}{n^2}$.)

Using a limit comparison test we see that
$\lim_{n\to\infty}\frac{\frac{(\ln(n))^3}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{(\ln(n))^3}{n}$ which after multiple applications of L'Hopital's Rule equals 0. Thus as $\Sigma \frac{1}{n^2}$ is a convergent p-series the given series converges as well.

- $$\Sigma_{n=1}^{\infty}\frac{1}{n\ln(n)}$$ should be 2

We apply the integral test.

Consider $\int_2^{\infty}\frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}\,dx$

let $u = \ln x$ then $du = \frac{1}{x}\,dx$

So we have

$$= \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{du}{u} = \lim_{b\to\infty} \ln|u|\Big|_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \ln(\ln b) - (\ln(\ln 2))$$

$$= \infty$$

$\Rightarrow$ Diverges by the integral test.

(OR apply the root test. $\lim \sqrt[n]{\frac{1}{\ln n}}$

- $$\Sigma_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$$

We apply the Root Test

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{n^{2n}}} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty.$$

By part (b) of the Root Test the series diverges.

- $$\Sigma_{n=1}^{\infty} \frac{n!}{n^n}$$

We apply the Ratio Test

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} (n+1) \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \Big\}$$ indeterminate form 1^∞

So let $y = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$ then

$$\ln y = \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}$$

$$\overset{\text{L.H.}}{=} \lim \frac{\left(\frac{n+1}{n}\right) \cdot \left(\frac{1}{(n+1)^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim -\frac{n}{n+1} = -1$$

So $y = \frac{1}{e} < 1 \Rightarrow$ convergence of series by ratio test

4. State and prove the Sandwich Theorem for Sequences.

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences. If

$a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then

$\lim_{n\to\infty} b_n = L$.

Prof.

By the definition of ~~limit~~ convergence we have

$$|a_n - L| < \varepsilon \iff L-\varepsilon < a_n < L+\varepsilon \quad \text{for } n > N$$

Likewise we have,

$$|c_n - L| < \varepsilon \iff L-\varepsilon < c_n < L+\varepsilon \quad \text{for } n > N$$

By the given we have, together with the above

$$L-\varepsilon < a_n \le b_n \le c_n < L+\varepsilon \quad \text{for } n > N$$

$$\Rightarrow \quad L-\varepsilon < b_n < L+\varepsilon \quad \text{for } n > N$$

$$\iff \quad |b_n - L| < \varepsilon \quad \text{for } n > N$$

$$\iff \quad \lim_{n\to\infty} b_n = L.$$

5. Find the series' radius and interval of convergence. For what values of x does the series converge (a) absolutely? (b) conditionally?

$$\Sigma_{n=\cancel{0}\ 2}^{\infty}(\ln(n))x^n$$

Apply the Ratio Test

$$\lim_{n\to\infty}\left|\frac{\ln(n+1)\cdot x^{n+1}}{\ln(n)\cdot x^n}\right| = |x|\cdot\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n)}$$

$$\overset{L.H.}{=} |x|\cdot\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}}$$

$$= |x|\cdot\lim_{n\to\infty}\frac{n}{n+1} = |x|\cdot 1$$

Thus we must have $|x|<1$ to have absolute convergence.

If $x = 1$ then we have

$$\sum_{n=2}^{\infty}\ln(n)\cdot 1$$ which diverges by the $n^{\underline{th}}$ term test for divergence

Therefore ~~by~~ we also consider $x=-1$ and the series diverges at this point also by the $n^{\underline{th}}$ term test for divergence.

Thus the series converges absolutely for $-1<x<1$.

It does not converge conditionally for any given x.

The radius of convergence is 1.

6. Find the Taylor polynomial of order 4 generated by $f(x) = \cos(x)$ at $a = 0$.

$f(x) = \cos x$ $\quad f(0) = 1$ $\quad a_0 = \frac{1}{0!} = \frac{1}{1}$

$f'(x) = -\sin x$ $\quad f'(0) = 0$ $\quad a_1 = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$ $\quad a_2 = \frac{-1}{2!}$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$ $\quad a_3 = 0$

$f^4(x) = \cos x$ $\quad f^4(0) = 1$ $\quad a_4 = \frac{1}{4!}$

$$\Rightarrow \quad P_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

This ~~third~~ forth order Taylor polynomial approximates $\cos(x)$. Use Taylor's Formula to estimate the error when $x = 1$.

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!} x^{n+1}$$

$f^5(x) = -\sin x$

On $(0,1)$ $-\sin x$ is bounded from above by 1 and below by -1.

Thus $R_4(1) = \frac{f^5(c) \cdot 1^{4+1}}{(4+1)!} = f^5(c) \cdot \frac{1}{5!}$

thus $-\frac{1}{5!} \le R_4(1) \le \frac{1}{5!}$