Calculus II - Exam 2 - Fall 2007

October 25, 2007

Name:
Honor Code Statement:

Directions: Justify all answers/solutions. Calculators are not permitted. You may use the table of trigonometric identities given on the last page.

1. Evaluate each of the following integrals.

   (a) $\int x \cos x dx$

Using integration by parts, we let

$u = x \qquad dv = \cos x\, dx$

$du = 1\, dx \qquad v = \sin x$

then $\int x \cos x dx = x \sin x - \int \sin x\, dx$

$= x \sin x + \cos x + C$

   (b) $\int_1^4 \sqrt{t} \ln t dt$

Using integration by parts we let

$u = \ln t \qquad dv = t^{1/2} dt$

$du = \frac{1}{t} dt \qquad v = \frac{2}{3} t^{3/2}.$

then $\int_1^4 \sqrt{t} \ln t = \frac{2}{3} t^{3/2} \ln t \Big|_1^4 - \frac{2}{3} \int_1^4 t^{1/2} dt$

$= \frac{2}{3} t^{3/2} \ln t - \frac{4}{9} t^{3/2} \Big|_1^4$

$= \frac{2}{3} t^{3/2} \left(\ln t - \frac{2}{3} \right) \Big|_1^4$

$= \frac{2}{3} \cdot 8 \left(\ln 4 - \frac{2}{3} \right) - \frac{2}{3} \left(-\frac{2}{3} \right)$

$= \frac{16}{3} \ln 4 - \frac{28}{9}$

(c) $\int e^{-\theta} \cos 2\theta d\theta$

Using integration by parts, we let

$$u = \cos 2\theta \qquad dv = e^{-\theta}\, d\theta$$
$$du = -2\sin 2\theta\, d\theta \qquad v = -e^{-\theta}$$

Then

$$\int e^{-\theta}\cos 2\theta\, d\theta = -e^{-\theta}\cos 2\theta - \int 2e^{-\theta}\sin 2\theta\, d\theta$$

To determine this integral we use integration by parts again.

We let

$$U = \sin 2\theta \qquad dV = +e^{-\theta}\, d\theta$$
$$dU = 2\cos 2\theta\, d\theta \qquad V = -e^{-\theta}$$

So we obtain that

$$\int e^{-\theta}\cos 2\theta\, d\theta = -e^{-\theta}\cos 2\theta - 2\left(-e^{-\theta}\sin 2\theta + 2\int +e^{-\theta}\cos 2\theta\, d\theta\right)$$
$$= -e^{-\theta}\cos 2\theta + 2e^{-\theta}\sin 2\theta - 4\int e^{-\theta}\cos 2\theta\, d\theta$$

And so

$$5\int e^{-\theta}\cos 2\theta\, d\theta = -e^{-\theta}\cos 2\theta + 2e^{-\theta}\sin 2\theta$$
$$\int e^{-\theta}\cos 2\theta\, d\theta = \frac{1}{5}\left(-e^{-\theta}\cos 2\theta\right) + \frac{2}{5}\left(e^{-\theta}\sin 2\theta\right) + C$$

(d) $\int (1-2\sin(2x))^2\,dx$

$$= \int 1 - 4\sin 2x + 4\sin^2 2x \, dx$$

$$= \int 1 - 4\sin 2x + 4\left(\frac{1-\cos 4x}{2}\right) dx$$

$$= \int 1 - 4\sin 2x + 2 - 2\cos 4x \, dx$$

$$= x + 2\cos 2x + 2x - \frac{1}{2}\sin 4x + C = 3x + 2\cos 2x - \frac{1}{2}\sin 4x + C$$

$$= 3x + 2(1-2\sin^2 x) - \frac{1}{2}\cdot 2\sin 2x \cos 2x + C$$

$$= 3x + 2 - 4\sin^2 x - (2\sin x\cos x)(1-2\sin^2 x) + C = 3x+2-4\sin^2 x - 2\sin x\cos x + 4\sin^3 x\cos x + C$$

(e) $\int \sin^3 7x\,dx$ You don't need the reduction formula to do this.

$$\int \sin^3 7x\,dx = \int \sin^2 7x \cdot \sin 7x\,dx$$

$$= \int (1-\cos^2 7x)\sin 7x\,dx$$

$$= \int \sin 7x\,dx + \int -\sin 7x\cos^2 7x\,dx$$

let $u = \cos 7x$

$du = -7\sin 7x\,dx$

$$= -\frac{1}{7}\cos 7x + \frac{1}{7}\frac{\cos^3 7x}{3} + C$$

(f) $\int \sin(3x)\cos(2x)\,dx$

Using a trig identity we obtain that

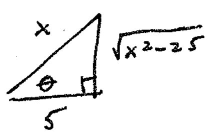
$$\int \sin 3x \cos 2x \, dx = \int \tfrac{1}{2}\sin x + \tfrac{1}{2}\sin 5x \, dx$$

$$= -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$$

(g) $\int \frac{\sqrt{x^2-25}}{x^4}\,dx$

Let $x = 5\sec\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

then $dx = 5\sec\theta\tan\theta\, d\theta$



Then $$\int \frac{\sqrt{x^2-25}}{x^4}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{625\sec^4\theta} \cdot 5\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\sqrt{25\tan^2\theta}\,\tan\theta\, d\theta}{125\sec^3\theta} = \frac{1}{25}\int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$$

$$= \frac{1}{25}\int \sin^2\theta\cos\theta\, d\theta = \frac{1}{25}\,\frac{\sin^3\theta}{3} + C$$

$$= \frac{1}{75}\left(\frac{\sqrt{x^2-25}}{x}\right)^3 + C$$

(h) $\int \frac{1}{(t+4)(t-1)}\,dt$

We first find the partial fraction decomposition

$$\frac{1}{(t+4)(t-1)} = \frac{A}{t+4} + \frac{B}{t-1}$$

Then

$$1 = A(t-1) + B(t+4)$$

If $t=1$ then $1 = 5B$ and so $B = \frac{1}{5}$

If $t=-4$ then $1 = -5A$ and so $A = \frac{-1}{5}$

Thus,

$$\int \frac{1}{(t+4)(t-1)}\,dt = \int \frac{-1/5}{t+4}\,dt + \int \frac{\frac{1}{5}}{t-1}\,dt$$

$$= -\frac{1}{5}\ln|t+4| + \frac{1}{5}\ln|t-1| + C$$

$$= \frac{1}{5}\left(\ln|t-1| - \ln|t+4|\right) + C$$

$$= \frac{1}{5}\ln\frac{|t-1|}{|t+4|} + C$$

2. Determine whether the integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_{-\infty}^{-1} e^{-2t}\,dt$ $= \lim_{\theta \to -\infty} \int_{\theta}^{-1} e^{-2t}\,dt$

$= \lim_{\theta \to -\infty} -\frac{1}{2} e^{-2t} \Big|_{\theta}^{1} = \lim_{\theta \to -\infty} -\frac{1}{2} e^{-2} + \frac{1}{2} e^{-2\theta}$

$= \infty$

This integral is divergent.

(b) $\int_{1}^{9} \frac{dx}{(x-9)^{1/3}}$

There is a discontinuity at $x = 9$

Thus we need to write

$$\lim_{t \to 9^-} \int_{1}^{t} \frac{dx}{(x-9)^{1/3}} = \lim_{t \to 9^-} \frac{3}{2}(x-9)^{2/3} \Big|_{1}^{t}$$

$$= \lim_{t \to 9^-} \frac{3}{2}(t-9)^{2/3} - \frac{3}{2}(-8)^{2/3}$$

$$= \frac{3}{2}(0) - 6 = -6$$

Thus, the integral is convergent.

3. If $\sum a_n$ and $\sum b_n$ are both divergent, is $\sum(a_n + b_n)$ necessarily divergent?

No. For example let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_1^{\infty} b_n = \sum_{n=1}^{\infty} \frac{-1}{n}$, each of which is divergent. Then $\sum_1^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} 0 = 0$ is a convergent series.

In more generality, let $\sum_1^{\infty} a_n$ be any divergent series and $\sum_1^{\infty} b_n = \sum_1^{\infty} -a_n$.

4. Determine whether the given sequence is increasing, decreasing, or not monotonic. Is the sequence bounded from above? from below? Justify your answers!

$$a_n = \frac{1}{5^n}$$

The sequence is decreasing, since $\frac{1}{5^{n+1}} < \frac{1}{5^n}$. Thus the sequence's first term is an upper bound. As $\frac{1}{5^n} > 0$ for all n, 0 is a lower bound.

The sequence is decreasing and bounded from below, thus...

Therefore by the Monotonic Sequence Theorem this sequence is . . . . . . convergent.

5. Determine whether the series is convergent or divergent. If it is convergent find its sum.

(i) $1 + 0.4 + 0.16 + 0.064 + \ldots.$

This is a geometric series with initial term $a = 1$ and common ratio $r = .4$. Thus as $|r| < 1$ the series converges to

$$\frac{1}{1-.4} = \frac{5}{3}$$

(ii) $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$

$$= \sum_{n=1}^{\infty} \left(\frac{-6}{5}\right)^{n-1}$$

This is a geometric series with common ratio $r = -\frac{6}{5}$. As $|r| > 1$ the series diverges.

6. Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{7n - n^{1/3}}{n^5}$$

$$= \sum_{n=1}^{\infty} \frac{7n}{n^5} - \frac{n^{1/3}}{n^5} = 7 \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{\infty} \frac{1}{n^{14/3}}$$

Each of these series is a convergent p-series, thus the given series is convergent.

5 7. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$$

The n^{th} term of the series is $\frac{n+2}{n^3+3n^2+3n+1}$.

For n large enough the n^{th} term is less than $\frac{1}{n^2}$

Since

$$\frac{n+2}{n^3+3n^2+3n+1} \le \frac{1}{n^2} \iff n^3+2n^2 \le n^3+3n^2+3n+1$$

$$\iff 0 \le n^2+3n+1$$

Holds since $n \ge 1$

Thus we can make a direct comparison

to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series.

Thus the given series converges.

5 8. Find all values of c for which the following series converges.

$$\sum_{n=1}^{\infty}\left(\frac{c}{n}-\frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \frac{(c-1)n + 1\cdot c}{n(n+1)}$$

We'll find a formula for the n^{th} partial sum.

$$S_n = \left(\frac{c}{1}-\frac{1}{2}\right)+\left(\frac{c}{2}-\frac{1}{3}\right)+ \cdots + \left(\frac{c}{n}-\frac{1}{n+1}\right)$$

$$= \frac{c}{1} + \frac{c-1}{2} + \frac{c-1}{3} + \cdots + \frac{c-1}{n} - \frac{1}{n+1}$$

$$= c + c-1 \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ \cdots + \frac{1}{n}\right) - \frac{1}{n+1}$$

$$= c + (c-1)\sum_{i=2}^{n}\frac{1}{i} - \frac{1}{n+1}$$

Thus

as $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} c + (c-1)\sum_{i=2}^{n}\frac{1}{i} - \frac{1}{n+1}$ and

The sum only exists when this limit exists.

This limit only exists if $c = 1$.

$\boxed{\text{OR}}$ Note that the series is

$$\sum_1 \frac{c-1}{n+1} + \sum \frac{c}{n(n+1)}.$$

the right hand series is convergent for any c, but the left hand only if $c=1$.