Calculus II - Exam 1 - Fall 2007

October 4, 2007

Name:

Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted. For your assistance, there is an attached sheet of trigonometric identities.

1. Calculate the following limits. Identify the indeterminate form (if any).
   
   \[ \lim_{x \to 1^+} \ln(x) \tan\left(\frac{\pi x}{2}\right) \]

   \[ \lim_{\theta \to 0^+} \frac{\ln(\theta)}{\theta} \]
• \( \lim_{x \to \infty} (e^x + x)^{\frac{1}{2}} \)

2. Find the second derivative of the following function:

\[ G(x) = \int_{3}^{x^2} 2^t \, dt \]
3. Differentiate the following function. Do not make use of a memorized formula, rather show how one can obtain the derivative directly by using Theorem 7 of Section 7.1, which states that if \( f \) is a one-to-one differentiable function with inverse function \( f^{-1} \) and \( f'(f^{-1}(a)) \neq 0 \), then the inverse function is differentiable at \( a \) and \( (f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} \).

\[ y = \tan^{-1}(x). \]
4. Evaluate the following integrals.

- \( \int 3\sin(x)\cos(x) \, dx \)

- \( \int \frac{\sec(x) \tan(x)}{2 + 3\sec(x)} \, dx \)
5. **Carbon 14** Carbon-14, one of the three isotopes of carbon, is radioactive and decays at a rate proportional to the amount present. Its half-life is 5730 years. If 10 grams were present originally, how much will be left after 2000 years. (As calculators are not permitted, you will leave your answer in terms of powers and logs.)
6. Define/State:

- State the Fundamental Theorem of Calculus, Part 2.

- Use logarithmic differentiation to find the derivative of the following function.

\[ y = \frac{x^{3/2} \sqrt{\sin(x) + 17x}}{x^3 - 37x} \]
7. Given below is a sketch of the proof of the expression of $e$ as a limit. For each of the FIVE starred * equality signs, give a justification as to why the step is valid (that is, give reason why what is on the left hand side of the equal sign is, in fact, equal to that which is on the right).

Given that if $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$. Thus $f'(1) = 1$.

We can thus say, that

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h} = \lim_{h \to 0} \frac{1}{h} \ln (1 + h)$$

$$= \lim_{h \to 0} \ln(1 + h) \frac{1}{h} = 1.$$

We then have that $\lim_{h \to 0} \ln(1 + h) \frac{1}{h} = 1$ and so we obtain that

$$e^1 = e^{\lim_{h \to 0} \ln(1 + h) \frac{1}{h}} = \lim_{h \to 0} e^{\ln(1 + h)^{1/h}} = \lim_{h \to 0} (1 + h)^{1/h}.$$