Calculus I - Final Exam
Fall 2008

December 9, 2008

Name:
Honor Code Statement:

100 points total.

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted, thus your solution to a problem need not be in decimal form. The point value of each problem is indicated in brackets.

1. [6 points] Find the inverse function and state its domain.

$$y = \ln(x+3)$$

Note that the natural logarithm function is one-to-one, so an inverse exists. So, we follow the procedure for finding an inverse as outlined on page 388 of text.

Solving for x in terms of y, we set

$$e^y = x + 3$$
$$e^y - 3 = x$$

Interchange x and y, we get.

$$y = e^x - 3$$

And, so $f^{-1}(x) = e^x - 3$. And the domain is $\mathbb{R}$ as the range of y is $\mathbb{R}$.

2. [3 points each] Give the following:

(a) the integral definition of the natural logarithmic function,

The natural logarithm function is the fct. defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt \quad , \quad x > 0 .$$

(b) the definition of e,

e is the number such that $\ln e = 1$, i.e

$$\int_1^e \frac{1}{t}\, dt = 1 .$$

(c) the derivative of the natural logarithmic function,

The derivative of $\ln x$ is $\frac{1}{x}$

(d) and the reason why its derivative is as such.

We know this is so by the Fundamental Theorem of Calculus, Part 1.

3. [5 points each] Differentiate each of the following functions:

(a) $f(x) = \sqrt{\ln(x)}$

$$f'(x) = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

(b) $g(x) = x\ln(x)$

$$g'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$$

(c) $y = \frac{e^x}{1+x}$

$$y' = \frac{(1+x)e^x - e^x}{(1+x)^2} = \frac{xe^x}{(1+x)^2}$$

(d) $g(t) = e^{\sin^2 t}$

$$g'(t) = e^{\sin^2 t} \cdot \frac{d(\sin^2 t)}{dt}$$

$$= e^{\sin^2 t} \cdot 2\sin t \cos t.$$

4. [6 points] Find an equation of the tangent line to the curve $y = 10^x$ at the point $(1, 10)$.

We will use the point-slope formula of line,

$$y - y_1 = m(x - x_1) \quad \text{where } (x_1, y_1) = (1, 10)$$

and m will be determined by computing $\left.\frac{dy}{dx}\right|_{x=1}$.

So,

$$\frac{dy}{dx} = 10^x \cdot \ln 10$$

and $\frac{dy}{dx} = 10^1 \ln 10$

Thus the equation of tangent line is.

$$y - 10 = 10 \ln 10 \, (x - 1)$$

$$y = 10 \ln 10 \, x - 10(\ln 10 - 1)$$

5. [5 points each] Evaluate each of the integrals:

(a) $\int \frac{(\ln(x))^2}{x} dx$

We use u-substitution method with $u = \ln x$, $du = \frac{1}{x} dx$.
Thus the integral becomes

$$\int u^2 \, du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C.$$

Replaced by

(a) $\int \frac{\cos x}{2+\sin x} dx$. We use u-substitution. And we have $\int \frac{du}{u} = \ln|u| + C$.
Let $u = 2+\sin x$
then $du = \cos x \, dx$
Thus the integral is $\ln|2+\sin x| + C$.

(b) $\int_0^5 e^{-3x} dx$. And, what theorem are you using to compute this?

We will use Fundamental Theorem of Calculus, Part II (The Evaluation Theorem).

$$\int_0^5 e^{-3x} dx = \frac{-1}{3}\int_0^5 -3e^{-3x} dx = \frac{-1}{3} e^{-3x}\Big|_0^5 = \frac{-1}{3} e^{-15} - \left(\frac{-1}{3} e^0\right)$$

$$= \frac{-1}{3} e^{-15} + \frac{1}{3}$$

(c) $\int \frac{\log_6(x)}{x} dx$

We will use the Change of Base formula to re-write

$\log_6 x = \frac{\ln x}{\ln 6}$.

The integral thus equals,

$$\int \frac{\ln x}{x} \cdot \frac{1}{\ln 6} dx = \frac{1}{\ln 6}\int \frac{\ln x}{x} dx.$$

Now again use a u-substitution with $u = \ln x$, $du = \frac{1}{x} dx$

and we obtain

$$\frac{1}{\ln 6}\int u \, du = \frac{1}{\ln 6}\frac{u^2}{2} + C = \frac{1}{\ln 6}\frac{(\ln x)^2}{2} + C.$$

6. [4 points each] Bismuth-210 has a half-life of 5 days.

(a) A sample originally has a mass of $800mg$. Find a formula for the mass remaining after t days.

We assume that decay occurs at a rate proportional to the amount present. So we adopt the model

$$y(t) = y_0 e^{kt}, \quad \text{where } y(t) \text{ is the mass of Bismuth-210 at time } t \text{ days}$$

We have 2 sample points $(0, 800)$ and $(5, 400)$. Thus

$$400 = 800 e^{5k} \Rightarrow \frac{1}{2} = e^{5k} \Rightarrow \frac{\ln(.5)}{5} = k.$$

Thus, $y(t) = 800 e^{\frac{1}{5}\ln(1/2) t} = 800 e^{\ln(1/2)^{t/5}} = 800 \left(\frac{1}{2}\right)^{t/5}$.

(b) Find the mass remaining after 30 days.

The mass after 30 days is

$$y(30) = 800 e^{1/5 \ln(1/2) \cdot 30} \text{ mg} = 800 e^{6 \ln(1/2)} \text{ mg}$$

(c) When is the mass reduced to $1mg$?

The mass is reduced to 1mg when

$$1 = 800 e^{1/5 \ln(1/2) t}$$

$$\frac{1}{800} = e^{1/5 \ln(1/2) t}$$

$$\ln(1/800) = \frac{1}{5} \ln(1/2) t \Rightarrow t = \frac{\ln(1/800)}{1/5 \ln(1/2)} \text{ days.}$$

(d) Give an expression for the average mass of the sample over the first 30 days.

We use the Mean Value Theorem for Integrals.

$$\text{Average mass} = \frac{1}{30} \int_0^{30} 800 e^{1/5 \ln(1/2) t} \, dt \text{ mg.}$$

7. [6 points] State the limit definition of the derivative and use this definition to find the derivative of $f(x) = 4x^2 - 10$.

The derivative of a function f at a number a, $f'(a)$, is $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ if this limit exists.

We use the definition to compute $f'(x)$ where $f(x) = 4x^2 - 10$.

$$f'(x) = \lim_{h \to 0} \frac{4(x+h)^2 - 10 - (4x^2 - 10)}{h}$$

$$= \lim_{h \to 0} \frac{4(x^2 + 2xh + h^2) - 10 - 4x^2 + 10}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2}{h} = \lim_{h \to 0} 8x + 4h.$$

$$= 8x.$$

8. [6 points] If $f(x) = x^3 - x$, does there exist a number c between 0 and 2 such that $f'(c) = \frac{f(2)-f(0)}{2}$? Why, or why not?

$f(x)$, a polynomial, is continuous and differentiable on all reals, and so specifically on $[0,2]$. Thus $f(x)$ meets the hypothesis of the Mean Value Theorem and so the conclusion holds, that is, there exists a $c \in [0,2]$ such that $f'(c) = \frac{f(2)-f(0)}{2-0}$.

9. [8 points] Find the x-coordinate of the point on the line $y = 2x + 2$ that is closest to the origin.

The distance from the origin to a point (x,y) on the line is given by:

$$D(x,y) = \sqrt{x^2 + y^2}.$$

We wish to minimize this function, but first we express it entirely in terms of x.

$$D(x) = \sqrt{x^2 + (2x+2)^2}$$
$$= \sqrt{5x^2 + 8x + 4}.$$

We compute

$$D'(x) = \frac{1}{2}(5x^2 + 8x + 4)^{-1/2} \cdot (10x + 8).$$

And set this to zero find a critical point at

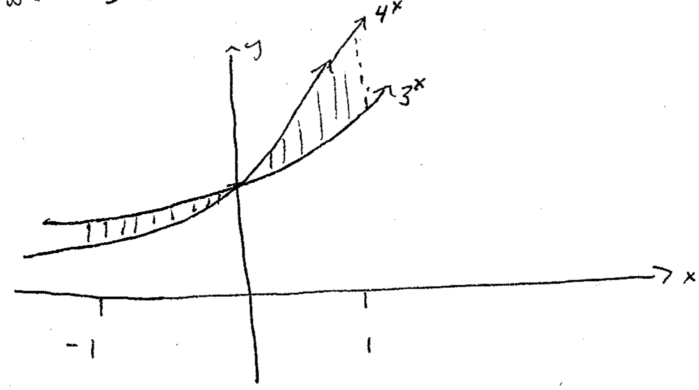
$$10x + 8 = 0$$
$$x = \frac{-4}{5}.$$

It is easy to check that this corresponds to a global minimum.

(This was similar to a homework problem, Section 4.7 #17.)

10. [5 points] Find the area of the region bounded by the curves $y = 3^x, y = 4^x, x = -1, x = 1$.

First we sketch the area



The area equals:

$$\int_{-1}^{0} 3^x - 4^x \, dx + \int_{0}^{1} 4^x - 3^x \, dx$$

$$= \frac{3^x}{\ln 3} - \frac{4^x}{\ln 4} \Big|_{-1}^{0} + \frac{4^x}{\ln 4} - \frac{3^x}{\ln 3} \Big|_{0}^{1}$$

$$= \left(\frac{3^0}{\ln 3} - \frac{4^0}{\ln 4}\right) - \left(\frac{3^{-1}}{\ln 3} - \frac{4^{-1}}{\ln 4}\right) + \left(\frac{4^1}{\ln 4} - \frac{3^1}{\ln 3}\right) - \left(\frac{4^0}{\ln 4} - \frac{3^0}{\ln 3}\right)$$

$$= \frac{2}{\ln 3} - \frac{2}{\ln 4} - \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{4}{\ln 4} - \frac{3}{\ln 3}$$

$$= \frac{1}{\ln 3}\left(2 - \frac{1}{3} - 3\right) + \frac{1}{\ln 4}\left(-2 + \frac{1}{4} + 4\right)$$

$$= \frac{-4}{3} \frac{1}{\ln 3} + \frac{9}{4} \frac{1}{\ln 4}$$