Calculus I - Exam 3
Fall 2008

November 18, 2008

Name:
Honor Code Statement:

85 points

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted. The point value of each problem is indicated in brackets.

1. [5 points] Find the derivative of the following function.

$$F(x) = \int_3^x \sin(t)\cos(t)dt$$

We apply the Fundamental Theorem of Calculus, Part 1, and obtain

$$F'(x) = \sin x \cos x$$

2. [5 points] Water flows from a tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.

We apply the Net Change Theorem

Amount of water lost from tank during 1st 10 minutes $= \int_0^{10} 200 - 4t \, dt = 200t - 2t^2 \Big|_0^{10}$

$= (200(10) - 2(100)) - (0)$

$= 1{,}800$ liters.

3. [5 points each] Evaluate the indefinite integral.

(a) $\int (5\cos x + \sec x \tan x + x^4)\, dx$

$$= 5\int \cos x \, dx + \int \sec x \tan x \, dx + \int x^4 \, dx$$

$$= 5\sin x + \sec x + \frac{x^5}{5} + C.$$

(b) $\int (1+\tan(\theta))^5 \sec^2(\theta)\, d\theta$

Using the substitution rule. Let $u = 1+\tan\theta$, and so $du = \sec^2\theta \, d\theta$

Thus this equals:

$$\int u^5 \, du = \frac{u^6}{6} + C$$

Now replacing u by $(1+\tan\theta)$ we obtain the indefinite integral $\frac{(1+\tan\theta)^6}{6} + C$

(c) $\int x^2 \sin(x^3)\, dx$

We use the substitution rule. We let $u = x^3$ and so $du = 3x^2\, dx$.

Thus,

$$\int x^2 \sin(x^3)\, dx = \frac{1}{3}\int 3x^2 \sin(x^3)\, dx$$

$$= \frac{1}{3}\int \sin u \, du$$

$$= -\frac{1}{3}\cos u + C$$

$$= -\frac{1}{3}\cos(x^3) + C.$$

4. [5 points] Why is the following statement false?

$$\int_{-2}^{1} \frac{1}{x^4}dx = \frac{-3}{8}$$

You might be tempted to write $\int_{-2}^{1} \frac{1}{x^4}\,dx = \frac{-1}{3}\frac{1}{x^3}\Big|_{-2}^{1} = \frac{-3}{8}$, but this would be wrong. We can't apply the Fundamental Theorem of Calculus Part II since $\frac{1}{x^4}$ is not continuous on the interval of integration.

Thus the statement is false.

5. [5 points] Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second.

(a) What is the meaning of $\int_{60}^{120} v(t)dt$?

This gives the change in position of the particle from $t = 60$ sec. to $t = 120$ sec.

(b) What is the meaning of $\int_{60}^{120} |v(t)|dt$?

It represents the total distance travelled by the particle from $t = 60$ sec to $t = 120$ sec.

6. [10 points] Sketch the region enclosed by $y = \sqrt{x}, y = \frac{1}{2}x$ and $x = 9$. Then find the area of the region.

See the sketch next page.

We find where $y = \sqrt{x}$ and $y = \frac{1}{2}x$ intersect.

$$\sqrt{x} = \frac{1}{2}x \iff x = \frac{1}{4}x^2 \iff 0 = \frac{1}{4}x^2 - x$$

$$0 = x\left(\frac{1}{4}x - 1\right)$$

When $x = 0$ and $\frac{1}{4}x - 1 = 0$

$x = 4$.

Thus area enclosed is

$$A = \int_0^4 \sqrt{x} - \frac{1}{2}x\,dx + \int_4^9 \frac{1}{2}x - \sqrt{x}\,dx$$

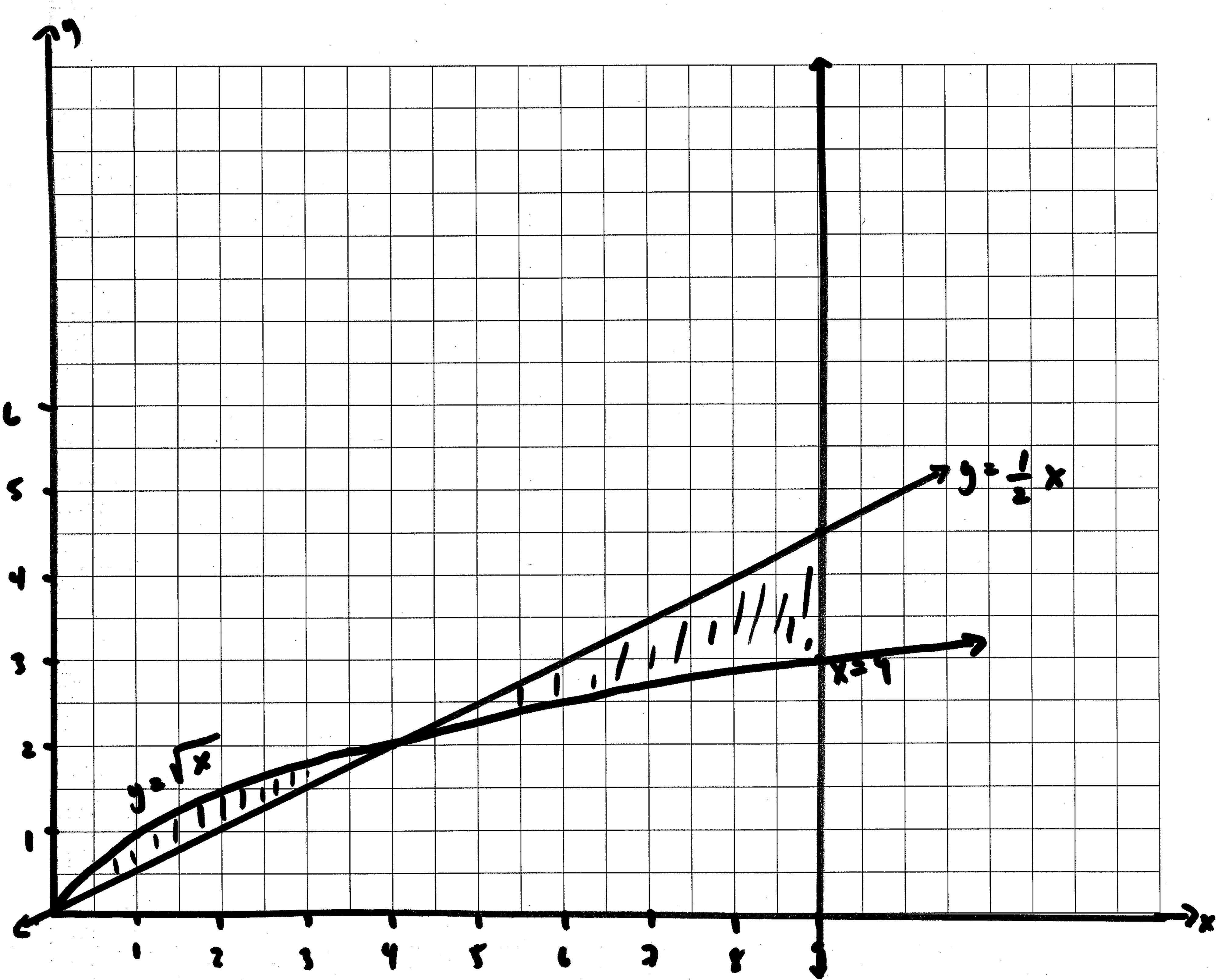
$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^2\Big|_0^4 + \frac{1}{4}x^2 - \frac{2}{3}x^{3/2}\Big|_4^9$$

$$= \left(\frac{2}{3}\cdot 8 - \frac{1}{4}\cdot 16\right) - (0) + \left(\frac{1}{4}\cdot 81 - \frac{2}{3}\cdot 27\right) - \left(\frac{1}{4}16 - \frac{2}{3}\cdot 8\right)$$

$$= \left(\frac{16}{3} - 4\right) + \left(\frac{81}{4} - 18\right) - \left(4 - \frac{16}{3}\right)$$

$$= \frac{32}{3} + \frac{81}{4} - 26$$

$$= \frac{128}{12} + \frac{243}{12} - \frac{312}{12} = \frac{59}{12}$$



y
x
$y = \sqrt{x}$
$y = \frac{1}{2}x$
$x = 9$
1
2
3
4
5
6
7
8
9
1
2
3
4
5
6

7. [10 points] Use the Principle of Mathematical Induction to prove that for each natural number n the following holds:

$$5 + 7 + 9 + \cdots + (2n + 3) = n(n + 4).$$

We prove the statement by induction.

We first establish the base case, i.e. when $n = 1$.

Note, $5 = 1 \cdot (1 + 4)$

And so the base case follows.

We now get to assume that the statement holds for $n = k$, that is, $5 + 7 + 9 + \ldots + (2k + 3) = k(k + 4)$. This is the inductive hypothesis.

We wish to show the statement holds for $n = k + 1$.

So we consider

$$5 + 7 + 9 + \ldots + (2k + 3) + (2(k+1) + 3)$$

$$= k(k + 4) + (2k + 5) \quad \text{, by the inductive hypothesis}$$

$$= k^2 + 6k + 5$$

$$= (k + 1)(k + 5).$$

The inductive step holds.

Thus, the statement is established for each natural number.

8. [10 points] Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

The average value of a function $f(x)$ over an interval $[a,b]$ is given by $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

We apply this here. We have:

$f_{ave} = 3$ $\quad f(x) = 2 + 6x - 3x^2$

$a = 0$

$b = b.$

Thus,

$$3 = \frac{1}{b}\int_0^b 2 + 6x - 3x^2\,dx$$

$$3 = \frac{1}{b}\left[(2x + 3x^2 - x^3)\Big|_0^b\right]$$

$$3 = 2 + 3b - b^2$$

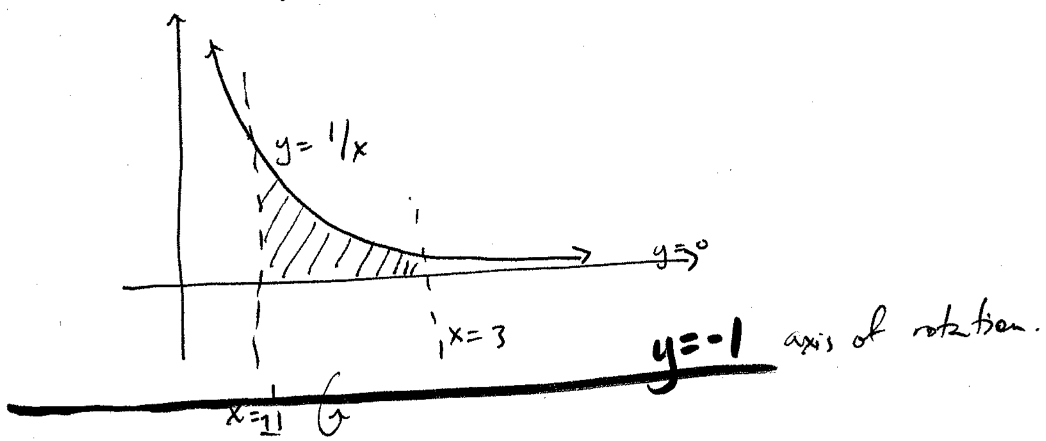
$b^2 - 3b + 1 = 0$ And we apply the quadratic formula to solve.

$$b = \frac{3 \pm \sqrt{9-4}}{2}$$

$$b = \frac{3 \pm \sqrt{5}}{2}.$$

9. [10 points] Find an expression for the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x}, y = 0, x = 1$ and $x = 3$ about $y = -1$. Sketch the region that is being rotated.

Sketch of region.



$$V = \pi \int_1^3 (r_{out})^2 - (r_{in})^2 \, dx$$

$$= \pi \int_1^3 \left(1 + \frac{1}{x}\right)^2 - 1^2 \, dx$$

$$= \pi \int_1^3 \frac{1}{x^2} + \frac{2}{x} \, dx$$