Calculus I - Exam 2
Fall 2008

October 28, 2008

70 points total.

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted. The point value of each problem is indicated in brackets.

1. [6 points] Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 3x + 1$ on $[0, 3]$.

We proceed using the Closed Interval Method.

Consider $f'(x) = 3x^2 - 3$ and set $f'(x) = 0$. to find critical points.

$0 = 3x^2 - 3 \iff x^2 = 1 \iff x = \pm 1$.

$+1$ is in the interval $[0, 3]$. So we consider $f(0) = 1$, $f(1) = -1$, $f(3) = 19$.

Thus we see that at 3 there is an absolute maximum with value 19, and at 1 there is an absolute minimum with value -1.

2. [6 points] What does it mean when we write $\lim_{x\to\infty} f(x) = L$?

It means that for every $\varepsilon > 0$ there exists a number N such that if $x > N$ then $|f(x) - L| < \varepsilon$.

(f) Determine any horizontal asymptotes of $f(x)$.

We consider $\lim_{x \to \infty} 1 - \frac{3}{x^2+3} = 1 - 0 = 1.$ } approaches from below

and $\lim_{x \to -\infty} 1 - \frac{3}{x^2+3} = 1 - 0 = 1$ } approaches from below.

Thus the horizontal asymptote is $y=1$.

(g) Determine any vertical asymptotes of $f(x)$.

Vertical asymptotes occur if and only if $x^2+3=0$.

But, $x^2+3=0$ has no real solutions.

(h) Determine intervals of increase and decrease.

We first find $f'(x) = 0 - \left[\frac{(x^2+3)\cdot 0 - 3(2x)}{(x^2+3)^2}\right] = \frac{6x}{(x^2+3)^2}.$

To find intervals of increase we ask when is $f'(x) > 0$.

$$\frac{6x}{(x^2+3)^2} > 0 \iff 6x > 0 \iff x > 0.$$

Thus $f(x)$ is increasing when $x>0$ and thus decreasing when $x<0$.

3. Consider the following function and find each of the following listed items. Upon completing each of the steps [2 points each], sketch the curve [6 points] on the graph paper provided.

$$f(x) = 1 - \frac{3}{x^2+3}$$

(a) Determine the domain of $f(x)$.

The domain is all real numbers.

(b) Determine the y-intercept of $f(x)$.

Let $x=0$ and we obtain $y = 1 - \frac{3}{0+3} = 1-1=0$.

(c) Determine any x-intercepts of $f(x)$.

Let $y=0$ and solve $0 = 1 - \frac{3}{x^2+3} = \frac{x^2+3-3}{x^2+3} = \frac{x^2}{x^2+3}$

$\Rightarrow x^2=0 \Rightarrow x=0$.

(d) Determine if the graph is symmetric about the y-axis.

We consider $f(-x) = 1 - \frac{3}{(-x)^2+3} = 1 - \frac{3}{x^2+3}$

As $f(-x) = f(x)$ there is symmetry about the y-axis.

(e) Determine if the graph is symmetric about the origin.

The graph is not symmetric about the origin as $f(-x) \neq -f(x)$.

(i) Determine any local extrema.

We have critical points when $f'(x)$ is undefined or equal to zero. Thus, as $f'(x)$ is always defined, we consider $0 = \frac{6x}{(x^2+3)^2} \Rightarrow 0 = 6x \Rightarrow x = 0.$

From the previous step we may conclude that at $x=0$ there is a local minimum.

(j) Determine intervals of concavity.

We consider the 2nd derivative.

$$f''(x) = \frac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3)2x}{(x^2+3)^4} = \frac{6(x^2+3)\left[(x^2+3) - 4x^2\right]}{(x^2+3)^4} = \frac{6(x^2+3)(-3x^2+3)}{(x^2+3)^4}$$

Note that $6>0$, $(x^2+3)>0$ and $(x^2+3)^4>0$ for all x.

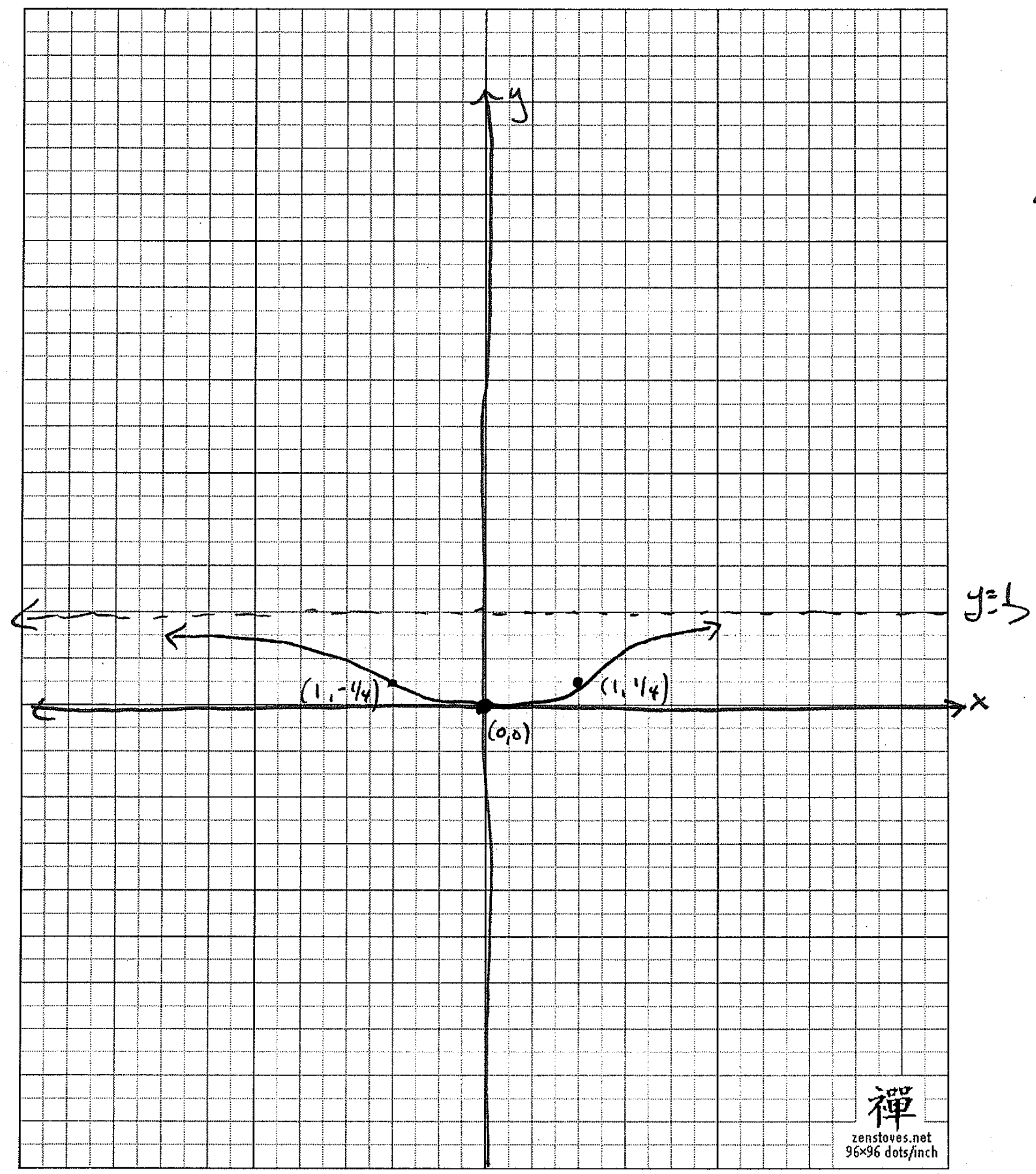
Thus $f''(x)>0$ when $(-3x^2+3)>0 \Leftrightarrow -3x^2>-3 \Leftrightarrow x^2<1 \Leftrightarrow -1<x<1$

Thus f is concave up on $(-1,1)$ and concave down otherwise.

(k) Determine any inflection points.

Inflection points may occur when $f''(x)=0$. This occurs when $-3x^2+3=0 \Leftrightarrow x=\pm 1$.

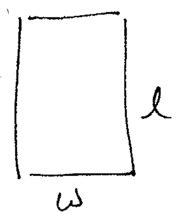
And these are inflection points $(1, \frac{1}{4})$ and $(-1, \frac{1}{4})$ as the previous step indicates a change in concavity at these points.



4 to 1 scale.

4. [12 points] Find the dimensions of a rectangle with area $1,000m^2$ whose perimeter is as small as possible.

Let the rectangle have width w and length l.



Denote the perimeter by P and the area by A.

We seek to optimize $P = 2l + 2w$.

We first express it as a single variable function.

We know $A = 1{,}000 = l \cdot w$. Thus $w = \frac{1{,}000}{l}$

So we may write $P = 2l + 2\left(\frac{1000}{l}\right)$.

We how compute the first derivative.

$$P' = 2 - \frac{2{,}000}{l^2}$$

We consider when $P' = 0$ to find the critical points. (Note that when $l=0$, P' is undefined, and this would correspond to a degenerate rectangle.)

$$0 = 2 - \frac{2{,}000}{l^2} \Rightarrow \frac{2000}{l^2} = 2 \Rightarrow l^2 = 1{,}000 \Rightarrow l = \sqrt{1{,}000}$$

Thus, $w = \sqrt{1{,}000}$ as $l \cdot w = 1{,}000$.

We check this corresponds to a minimum. We use the 2nd derivative test.

$P'' = +\frac{4{,}000}{l^3}$. Note $P''(10\sqrt{10}) > 0 \Rightarrow$ the point corresponds to a global minimum.

5. [6 points] Find all numbers c that satisfy the conclusion of the Mean Value Theorem when $f(x) = x^3 + x - 1$ and we consider the interval $[0, 2]$.

$f(x)$ is a continuous and differentiable function for all real numbers. As it satisfies the hypothesis of MVT, we may claim the conclusion. That is, there exists a $c \in [0,2]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We may find the specific value of c.

$f'(x) = 3x^2 + 1$

$f(2) = 9$
$f(0) = -1$ and $\frac{f(2)-f(0)}{2-0} = \frac{9-(-1)}{2-0} = \frac{10}{2} = 5$

So we ask when does $f'(x) = 5$?

$3x^2 + 1 = 5 \iff 3x^2 = 4 \iff x^2 = \frac{4}{3} \iff x = \pm\sqrt{\frac{4}{3}}$

Note that $\sqrt{\frac{4}{3}}$ is in $[0,2]$.

6. Given below is an incomplete statement and sketch of a particular case in the proof of Rolle's theorem. Your task is to complete the sketch by filling in the blanks. There are 6 blanks to fill in [2 points each].

Theorem 1 *[Rolle's Theorem] Let f be a function that satisfies the following three hypotheses:*

(a) f is continuous on the closed interval $[a, b]$.
(b) f is differentiable *on the open interval (a, b).*
(c) $f(a) =$ $f(b)$

Then there is a number c in (a, b) such that ....

PROOF OF THE CASE $f(x) > f(a)$ FOR SOME $x \in (a, b)$:
By the Extreme Value Theorem (which we can apply by the first hypothesis), f has a maximum value somewhere in $[a, b]$. By the third hypothesis, it must attain this maximum value at a number c in the open interval (a, b). Then f has a local max at c and, by the second hypothesis, f is differentiable at c. Therefore, the conclusion holds by Fermat's Theorem.

OMIT.

7. [4 points] Find the most general antiderivative of the function.

$$f(x) = 8x^9 - 3x^6 + 12x^3 - \cos x$$

$$F(x) = \frac{8}{10}x^{10} - \frac{3}{7}x^7 + 3x^4 - \sin x + C$$

for some constant C.