Calculus I - Exam 1
Fall 2008

October 7, 2008

Name:
Honor Code Statement:

83 points.

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted. The point value of each problem is indicated in brackets.

1. [5 points each] Find the derivative of each of the following functions. You may use any method you wish. [1 point each] **STATE** the rule or method you use for each problem.

   (a) $y = x^3 - 10x^2 + 17$

   We use the power rule to find,

   $$y' = 3x^2 - 20x$$

   (b) $g(t) = \sqrt{(sin(t))}$

   We use the chain rule (and power rule) to find,

   $$g'(t) = \frac{1}{2}(\sin t)^{-1/2} \cdot \cos t$$

   $$= \frac{\cos t}{2\sqrt{\sin t}}$$

(c) $F(x) = x^2 \cos x$

We use the product rule to find.

$$F'(x) = x^2(-\sin x) + \cos x \cdot 2x$$

$$= x(2\cos x - x\sin x)$$

(d) $xy^4 + x^2y = x + 3y$

We use implicit differentiation to find

$$x\,4y^3y' + y^4 + x^2y' + y\,2x = 1 + 3y'$$

And so,

$$x\,4y^3y' + x^2y' - 3y' = 1 - y^4 - y\,2x$$

$$y'(x\,4y^3 + x^2 - 3) = 1 - y^4 - 2xy$$

$$y' = \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$$

2. [8 points] Find an equation of the tangent line to the curve at the given point.

$$y = \frac{x^2+1}{x^2-1}, \quad (0, -1)$$

We use the quotient rule to find y'.

$$y' = \frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2} = \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

At $x=0$ $\quad y'(0) = \frac{-0}{1^2} = 0$

Thus, the equation of the tangent line, using the point-slope formula of a line, is

$$y-(-1) = 0(x-0) \iff y+1=0 \iff y=-1$$

[3 points] Does this function have any discontinuities? If so, where? If so, what type?

Yes. The function has discontinuities when $x^2-1=0$, i.e at $x=\pm 1$.

These are infinite discontinuities.

3. [8 points] If a stone is thrown vertically upward from the surface of the moon with a velocity of $10m/s$, its height (in meters) after t seconds is $h = 10t - 0.83t^2$. What is the velocity of the stone after $3s$?

As h gives position, we find h' to find the velocity.

$$h' = 10 - 2(.83)t$$

Thus the velocity after $3s$ is

$$h'(3) = 10 - 2(.83)3$$
$$= 5.02 \ m/s$$

4. [6 points] Is there a root of the equation in the given interval? Why, or why not?

$$2x^3 + x^2 + 2 = 0, \qquad (-2, -1)$$

Let $f(x) = 2x^3 + x^2 + 2$.

Then $f(-2) = 2(-2)^3 + (-2)^2 + 2 = -10$

$f(-1) = 2(-1)^3 + (-1)^2 + 2 = 1$

Note that as $f(x)$ is a polynomial, $f(x)$ is continuous everywhere.

The hypothesis of the Intermediate Value Theorem is satisfied, thus we claim the conclusion. That is,

there is a root to the above equation, i.e. there exists a $c \in (-2,-1)$ such that $f(c) = 0$.

5. [4 points] Does $\lim_{x\to a} \cos(x + \sin x) = \cos(\lim_{x\to a} (x + \sin x))$? Why, or why not?

Yes, the limits are equal. As $\cos x$ is continuous everywhere and $\lim_{x\to a} (x + \sin x)$ exists for all reals, then we may apply Theorem 8 of Section 2.5.

6. [8 points] Prove that $\lim_{x\to 3}(6x - 2) = 16$ using the ϵ, δ definition of limit.

We say that $\lim_{x\to a} f(x) = L$ if for every ε there exists a δ such that $|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$.

So, we ask when is $|(6x-2) - 16| < \varepsilon$?

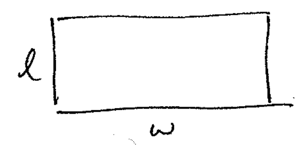
$|6x-2-16| < \varepsilon$ whenever $|6x - 18| < \varepsilon$

whenever $6|x-3| < \varepsilon$ whenever $|x-3| < \frac{\varepsilon}{6}$.

Thus, if we let $\delta = \frac{\varepsilon}{6}$ then $|6x-2-16| < \varepsilon$ whenever $|x-3| < \delta$.

7. [8 points] The length of a rectangle is increasing at a rate of $8cm/s$ and its width is increasing at rate of $3cm/s$. When the length is $20cm$ and the width is $10cm$, how fast is the area of the rectangle increasing?

We first consider a diagram.



Let l = length in cm.
w = width. in cm
t = time in s
A = area of rectangle in cm^2

We know

$$A = l \cdot w \quad , \quad \frac{dl}{dt} = 8cm/s \quad , \quad \frac{dw}{dt} = 3\frac{cm}{s}.$$

If we want $\frac{dA}{dt}$ we must differentiate. using implicit differentiation and the product rule

So,

$$\frac{dA}{dt} = l \cdot \frac{dw}{dt} + w \cdot \frac{dl}{dt}$$

Evaluating at $l = 20$, $w = 10$, we obtain

$$\frac{dA}{dt} = 20 \cdot 3 + 10 \cdot 8 = 140 \ cm^2/s.$$

8. Given below is an incomplete sketch of the proof of the following theorem: *If f is differentiable at a, the f is continuous at a.* Your task is to complete the sketch. For each of the **starred** * equality signs [2 points each], give a justification as to why the step is valid (that is, give reason why what is on the previous side of the equal sign is, in fact, equal to that which follows). There are two blanks to fill in [2 points each].

PROOF:

To show that f is continuous at a, we must, by the definition, show that $\lim_{x\to a} f(x)$ equals ...f(a)... (Fill in the blank.)

Thus, we consider

$\lim_{x\to a} f(x) = \lim_{x\to a} f(a) + f(x) - f(a)$,

by a clever trick of adding zero,

$=^* \lim_{x\to a} f(a) + \lim_{x\to a} f(x) - f(a)$,

as the limit of a sum is the sum of the limits.

$=^* \lim_{x\to a} f(a) + \lim_{x\to a} [(f(x) - f(a))\frac{x-a}{x-a}]$,

by a clever trick of multiplying by 1.

$=^* \lim_{x\to a} f(a) + \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a} x - a$,

as the limit of a product is the product of the limits.

$=^* \lim_{x\to a} f(a) + f'(a)\cdot 0,$

as $f'(a)$ exists by the hypothesis and $\lim_{x\to a} x-a = 0$ as $x-a$ is a polynomial.

$=^* \lim_{x\to a} f(a),$

Multiplication by zero.

$= \underline{f(a)}$ Fill in the blank.

This completes the proof.