MATH 704: Senior Seminar
Fall 2010

September 6, 2010

Contents

1 Catalog Description

MATH 0704 Senior Seminar (Fall, Spring)

Each student is required to complete and present a major paper on a topic chosen with the advice of a faculty member. In addition, during the academic year, each student is expected to attend a series of lectures designed to introduce and integrate ideas of mathematics not covered in the previous three years.

2 Course Supervisor

John Schmitt
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E-mail: jschmitt@middlebury.edu
My (and course) webpage: http://community.middlebury.edu/∼jschmitt/
Office Hours: Monday and Friday 11am—12pm, Thursday 10am—12pm, or by arrangement

3 Purposes of the Seminar

The Senior Seminar in Mathematics has two primary purposes:

1. First, to expose mathematics majors to a number of important and interesting topics in the mathematical sciences that are normally not presented within the department's set of required courses. This goal is accomplished by the series of talks given by visiting mathematicians, members of the Middle bury faculty, and the senior majors.

2. Second, to give majors the opportunity to explore in depth a topic in pure or applied mathematics. This experience textitasizes some aspects of education not normally stressed in our regular courses: independent study, library research, organizing and internalizing a chunk of mathematics, expository writing, and verbal presentation of material. The Senior Seminar requires that each enrolled student participate in an active fashion. **ATTENDANCE AT ALL THE SCHEDULED SEMINAR TALKS, LECTURES AND FILMS DURING THE STUDENT'S SENIOR YEAR IS DEMANDED: ACTIVE PARTICIPATION IS EXPECTED.** We ask all our speakers to gear their talks to an undergraduate audience so that all student listeners can make sense of what is going on and can learn some mathematics from the presentations.

The major part of the Seminar for each member, however, is the senior thesis. Each year the members of the department prepare a list of some possible senior thesis topics, organized by general area. Each student selects a topic under the guidance of a member of the faculty. Students who desire to study a topic not on that list may do so provided they have an agreement from a faculty member to serve as a supervisor. Either way, students must have a topic chosen and approved by a faculty advisor prior to registration. The current list of topic suggestions appears below.

4 Schedule

The following schedule of deadlines will be in effect for Fall Semester, 2010:

Week 1: Inform me of your thesis topic and faculty adviser, and find at least one core reference. Register for the course, if you have not already done so. E-mail details to MATH 704 adviser (jschmitt@middlebury.edu) by 5 PM on **Friday, September 10**. Install the appropriate version of LaTeX on your computer.

Week 2: Give a 5-minute presentation in class on your thesis topic. **(Thursday, September 16)**

Week 3: Submit to thesis adviser and MATH 704 supervisor a bibliography of books, papers, and other references. **(Thursday, September 23)**

Week 5: Give a 10-minute presentation on your thesis topic **(Tuesday, October 5 and Thursday, October 7)** and submit outline of thesis to both advisers;

Week 6: Give oral update on status of research and writing **(Thursday, October 14)**

Week 8: Submit first draft of thesis to thesis adviser **(Thursday, October 28)**

Week 10: Submit second draft of thesis to thesis adviser **(Thursday, November 11)**

Week 12: Eat a turkey, or portion thereof.

Week 13: Submit final draft of thesis to thesis adviser **(Monday, November 29)**

Week 13: Presentation of thesis talks **(Tuesday, November 30, through Thursday, December 2)**

Week 13: Submit polished final thesis to MATH 704 supervisor by 5 PM on **Monday, December 6**. You should hand in three printed copies and one digital version(disk or CD). **Late Penalties: The department's policy normally requires a minimum penalty of a drop of one grade for each day, or portion thereof, the thesis is late. Thus a thesis which would have been awarded an A based on quality, for example, would receive an A- if the final version is submitted between 5 PM Monday and 5 PM Tuesday, a B+ if submitted between 5 PM Tuesday and 5 PM Wednesday, a B if submitted between 5 PM Wednesday and 5 PM Thursday.**

In addition to weekly class meetings, it is the student's responsibility to meet with the faculty adviser at least once a week.

You should begin with the realization that the senior seminar in mathematics is supposed to be the capstone of your educational experience at Middlebury. It should be your number one priority for the term. In the second half of the semester, large blocks of your time will be devoted to writing your paper.

This schedule requires the student to make an early commitment to a particular topic and to begin work on that subject at the start of the term. You can eliminate some of the last minute all-nighters and panicky final days by disciplined work in the first half of the term. Set a definite period of time each day which you will devote to your thesis. Arrange regular meetings with your adviser at least once a week.

The bibliography, thesis outline, and the first and second drafts will normally be evaluated and returned to the student by the thesis supervisor within one week. The thesis supervisor will read the final draft and indicate corrections which must be made before the polished final thesis is submitted. **The evaluation of how well these course requirements were met will be considered by the Department Faculty in determining the final grade for the course.**

Students sometimes experience difficulty in their first independent learning experience. The structure imposed on you in regular courses – classes meeting two or three times a week, daily homework, scheduled examinations – makes it somewhat easier for you to organize your time and to discipline yourself to meet deadlines. Some of that structure is (deliberately) missing from the Senior Seminar. You will have to take more initiative in organizing your schedule to complete the work on your thesis.

5 Plagiarism

The habit of intellectual honesty is essential to both intellectual and moral growth. Effective evaluation of student work and helpful instruction can take place only in an environment where intellectual honesty is respected.

The relevant *Middlebury College Handbook* language is as follows:

"As an academic community devoted to the life of the mind, Middlebury College requires of every student complete intellectual honesty in the preparation of all assigned academic work.."

"Plagiarism is a violation of intellectual honesty. Plagiarism is passing off another person's work as one's own. It is taking and presenting as one's own the ideas, research, writings, creations, or inventions of another. It makes no difference whether the source is a student or a professional in some field. For example, in written work, whenever as much as a sentence or key phrase is taken from the work of another without specific citation of the source, the issue of plagiarism arises.

"Paraphrasing is the close restatement of another's idea using approximately the language of the original. Paraphrasing without acknowledgment of authorship is also plagiarism and is as serious a violation as an unacknowledged quotation...

"The individual student is responsible for ensuring that his or her work does not involve plagiarism. Ignorance of the nature of plagiarism or of College rules may not be offered as a mitigating circumstance." [*Middlebury College Handbook 2002-2003*, page 105]

As many theses in our department involve restatements of known theorems and the proofs of the same, the question of plagiarism may be relevant to you. Your thesis adviser will be able to answer any questions you may have about this subject. It is your responsibility to consult with him/her.

6 Texts

There are no required texts for the seminar, but we recommend *On Writing Well*, 6th Ed., by William Zinsser as a good general guide to expository writing and the always indispensable *The Elements of Style* by William Strunk and E. B. White.

There are also several excellent short books on the writing of mathematics. These include:

- *How To Write Mathematics* by Norman Steenrod, Paul Halmos, Menahem Schiffer, and Jean Dieudonn,
- *A Primer of Mathematical Writing* by Steven G. Krantz,
- *Mathematical Writing*, edited by by Donald E. Knuth, Tracy Larrabee, and Paul M. Roberts, and
- *Handbook of Writing for the Mathematical Sciences* by Nicholas J. Higham.

A copy of these books will be available on reserve in the main Library. Some reading assignments from these books may be made.

7 Library Assistance

The College's reference librarians can be of enormous assistance to you in assisting you to identify and locate relevant materials whether in printed or digital format. Contact Bryan Carson (extension 5341), who is the science librarian most familiar with mathematics and computer science materials.

You may also apply at the Circulation Desk at the Library for special senior thesis priviliges, such as extended checkout periods, a carrell, or a locker.

8 Thesis Expectations

Theses that earn good grades have some of the following qualities in common:

1. *Well presented*. The thesis should have a minimum of typographical errors and misspelled words and be neat and evenly spaced. Take the time to run a spell checker and to proofread. Spelling and grammar do count. Have a friend proofread it for you; new eyes will see mistakes you've missed. Even if the content is great, you won't get a top grade unless it is also presented in a readable, comprehensible manner.

2. *Relatively difficult mathematics*. You must learn something new, something you did not see in a class, and something with substance. The work does not have to be very broad, but it should be somewhat deep. One option is to take a narrow topic and learn a lot about it. You should understand the topic and be able to explain it in your own words.

3. *Independent work.* You should do most of the learning on your own. Read the material; try to work through proofs by yourself. Feel free to ask questions of your adviser, but do not expect your adviser to present the material to you or to do the proofs for you. If you get stuck, it is perfectly acceptable to ask for hints or for you and your adviser to work through a problem together. Ideally you should combine material from several sources and draw your own conclusions or arrange the material in an original way.

For example, you might pose a problem. Then you could try to work out your own solution (or examples) and/or find several different solutions (or examples) from several different sources. You might discuss the similarities and differences of these solutions: Do they use different types of mathematics? Do you use both number theory and geometry? Can you draw your own illustrations or examples of these methods? Insert some of yourself into your thesis – your opinions, your arrangement of material, your own proofs, or your examples.

4. *Correctness (or accountability).* Check your definitions; be sure that your theorems are correct and that your proofs make sense to you. Can you explain them in your own words? Be very careful that you're not just mimicking someone else's proof and that you really do understand the words you are using.

There is a fine line here between doing your own work and plagiarizing someone else's. If you copy something word for word without using quotation marks (or setting it off in a narrow paragraph) and including a citation, that is plagiarism. Using well established definitions from the literature is not plagiarism, but you must acknowledge your source. It is not acceptable to string together paragraph after paragraph of quoted material. You should be doing most of the writing yourself, using quotations to support a point.

For the expository parts of your thesis, gather the information and then express the ideas in your own words. The definitions can be quoted. The proofs should be, as much as possible, your own. That doesn't mean that you have to prove everything yourself. Working through someone else's proof is a perfectly acceptable thing to do. However, when it comes time to write up your thesis, you should try, as much as possible, to express the ideas in your own words.

5. *Parts completed on schedule.* This means not only meeting all intermediate deadlines, but also holding regular (usually weekly) meetings with your adviser. This is not meant to be a last minute, night before project. This is meant to be a semester long research project. Get each part done on time! Get regular feedback from your adviser. You will need the time available to absorb and understand the difficult concepts. It you understand all of the ideas the first time through, then your problem (project, topic) may not be hard enough.

6. *Appropriate depth and length* Your thesis should delve deeply enough into the subject area that your analysis requires reasonably sophisticated undergraduate mathematics. Your treatment of the material should be sufficiently extensive to explore the topic thoroughly ad carefully, but do not overdo the length. Most theses are between 40 and 60 pages in length. Theses that run longer than 60 pages tend to be hurriedly and awkwardly written. You will need to get permission for your thesis advisor to go beyond 60 pages. The department has set an absolute maximum length of 100 pages for a senior thesis. Your goal

should be **Quality**, not **Quantity**.

9 Basis for Evaluation

In many departments of the College, a senior thesis is an option available for students seeking Honors in the majors. In our program, all majors are required to complete a one term senior thesis. The distribution of grades in MATH 704 has followed a pattern quite similar to the distribution of grades in other senior level courses in the department. For your information, here is a distribution of MATH 704 grades from 1975 through Spring 2010 (note that A+ is no longer an available grade):

Grade	Number	Grade	Number
A+	12	C+	34
A	91	C	16
A-	117	C-	9
B+	125	D	9
B	86	F	1
B-	53		

Keep in mind that each year the department awards the Dr. Francis D. Parker, '39 Mathematics Prize. The award was established in 1993 by Dr. Francis D. Parker, class of 1939. It is awarded to the graduating student for the best senior work in the mathematics department and is largely based upon the senior thesis, though other accomplishments (Putnam scores, Green Chicken scores, conference participation, etc.) are considered as well.

10 Typsetting Your Thesis

After a highly successful experiment in Fall 2005, the Mathematics Department decided to require that all seniors compose their theses using TEX. TEX is a typesetting system created initially in the late 1970s by Stanford mathematician and computer scientist Donald Knuth. Knuth designed TEX to allow any individual to produce high-quality typeset books and articles using a reasonable amount of effort, and to provide a system that would give the exact same results on all computers, now and in the future.

TEX is generally considered to be the best way to typeset complex mathematical formulas, but, especially in the form of LATEX and other template packages, is now also being used for many other typesetting tasks. LATEX offers programmable desktop publishing features and extensive facilities for automating most aspects of typesetting and desktop publishing, including numbering and cross-referencing, tables and figures, page layout, bibliographies, and much more. LATEX was originally written in 1984 by Leslie Lamport and has become the dominant method for using TEX . Few people write in plain TEX anymore.

Software packages that incorporate LaTeX and TeX are widely available on the web and are free of charge. We recommend MikTeX for Windows users and TeXShop for Macintosh users. We will distribute CD's with the requisite software or provide detailed directions on finding and downloading what you need to install.

Several books on using LaTeX have been put on one-day reserve at the library. They are:

- *Learning LaTeX* by David Griffiths and Desmond Higham. I highly recommend you purchase a copy.
- *A Guide to LaTeX: Document Preparation for Beginners and Advanced Users* by Helmut Kopka and Patrick Daly.
- *The LaTeX Companion* by Michel Gossens, Frank Mittelbach, and Alexander Samarin.
- *First steps in LaTeX* by George Grätzer.
- *LaTeX : a document preparation system: user's guide and reference manual* by Leslie Lamport.

The introduction to LaTeX by Griffiths and Higham is a particularly friendly guide. It is short, engagingly written with many examples showing side-by-side the LaTeX code and the final typeset text. We will distribute excerpts from this book for your use. A preliminary edition is available in pdf format at
$http://www.maths.dundee.ac.uk/ftp/na-reports/CS01_LaTeXGuide.pdf$
and also in the Handouts folder of on the classes server.

In the Handouts folder there is a more complete guide to LaTeX, titled *A Not So Short Guide to LaTeX* by Tobias Oetiker et al.

Professor Swenton has prepared several templates for use with LaTeX that will make it easy for you to begin composing your thesis. These are contained in the folder mcthesis which is the Handout folder.

11 Topics

These are ideas that various faculty members have suggested for thesis topics over the years. The list is by no means exhaustive and is somewhat dated. If you have a potential adviser in mind, that person will have additional suggestions. Or you may even have your own idea for a project. We encourage this route as well, but please be aware that this will put some additional responsibility on you to identify sources.

Note: Steve Abbott and Frank Swenton are on leave for the academic year.

1. **Inversive Geometry**

   The geometry of the plane in which figures are equivalent when one is the "inverse" of another with respect to a fixed circle has a long history.

It is an interesting non-Euclidean geometry in its own right, a link with classical projective geometry and the mechanism for proof of many otherwise difficult Euclidean theorems including Feuerbach's Theorem on the 9-point circle.

For further information, see Bruce Peterson.

2. **The Four Color Theorem**

   For many years, perhaps the most famous unsolved problem in mathematics asked whether every possible map on the surface of a sphere could be colored in such a way that any two adjacent countries were distinguishable using only four colors. It is easy to produce maps requiring at least four colors, but the proof that four colors are always sufficient did not appear until 1976. Topics for a thesis would include the history of the problem, including the mistakes made in early "proofs", extension of the problem to more complicated surfaces (what for instance happens if the maps are drawn on the surface of an inner tube?), and an explication of the final correct proof. The proof itself marks a milestone in mathematics in that it is readily understandable, but impossible to check because it involves computer verification of an enormous number of special cases. That is, anyone can check any individual step, but no one can check them all. The thesis would not involve computer work.

   Reference: Sandy Hunt, "The Four Color Theorem," Senior Thesis, 1987.

   Jennifer Paris, "The Four Color Problem," Senior Thesis, 1991.

   For additional information, see Bruce Peterson.

3. **Additive Number Theory**

   We know a good deal about the multiplicative properties of the integers – for example, every integer has a unique prime decomposition. Less is known about how to decompose integers additively. For instance, in how many ways can we write an integer as the sum of two squares? of four squares? How many ways can we write the number 1 as the sum of three cubes? Is every number the sum of two primes (Goldbach's conjecture)? What is the relationship between the divisors of a number and its partitions (additive decompositions)?

   For related ideas, see Waring's Problem and Additive Bases.

   Reference: Hardy and Wright, *An Introduction to the Theory of Numbers*

   Peter Schumer, *Introduction to Number Theory*

   For further information, see Peter Schumer, David Dorman, or Priscilla Bremser.

4. **Fermat's Last Theorem**

   About 1631, Pierre de Fermat claimed to have proved that the equation $x^n + y^n = z^n$ had no solutions in integers except in the special cases

$n = 1$ and $n = 2$. He claimed that there are no integers x,y,z such that $x^3 + y^3 = z^3$, and none if the exponent is 4, 5, etc. This theorem has recently been proved by Andrew Wiles of Princeton University. While a discussion of his proof is well beyond the undergraduate level a thesis could include proofs of several of special cases, a discussion of quadratic number fields and norms (material intersecting with and on the same level as MATH 302), and perhaps a discussion of Bernoulli numbers, irregular primes and Kummer's Theorem, which establishes a large class of numbers for which Fermat's Last Theorem is known to be true.

References: Thomas Nuovo, "Fermat's Last Theorem", Senior Thesis 1985.

Paul Ribenbaum, *13 Lectures on Fermat's Last Theorem*, Springer-Verlag, 1979.

For further information, see Peter Schumer, David Dorman, or Priscilla Bremser.

5. **Mersenne Primes and Perfect Numbers**

   Numbers like 6 and 28 were called *perfect* by Greek mathematicians and numerologists since they are equal to the sum of their proper divisors (e.g., $6 = 1 + 2 + 3$). Euclid showed that if $2^p - 1$ is a prime number then $n = 2^{p-1}(2p - 1)$ is perfect. Since then (about 300 B.C.) there has been a great deal of interest in finding large perfect numbers. Consequently, some mathematicians have tried to determine the values of p for which $2^p - 1$ is prime. Such primes are called *Mersenne primes* after Friar Marin Mersenne who in 1644 conjectured all values of $p \leq 257$ for which $2^p - 1$ was believed to be prime. There still remain many open questions, for example, do there exist any odd perfect numbers? This would make a very nice thesis topic for anyone wishing an introduction to number theory. Research could include some interesting computer work if desired.

   Reference: T. McCoy, "Mersenne Primes and Percent Numbers", Senior Thesis, 1987.

   J. Paris, "The Generation of Amicable Numbers", Senior Thesis, 1985

   For further information, see Peter Schumer, David Dorman, or Priscilla Bremser.

6. **Group Decision-Making**

   Arrow's Impossibility Theorem has generated a great deal of research on developing mathematical models of individual and group decision-making. Recent results indicate that any "reasonable" voting procedure must either be dictatorial or subject to strategic manipulation. Many "possibility" theorems have been proved for voting mechanisms which satisfy relaxed versions of Arrow's axioms.

   References: Kelley, *Arrow Impossibility Theorems.*

   For further information, see Michael Olinick.

7. **Logistic Models of Population Growth**

   The classic logistic model for the growth of a population P over time t is $dP/dt = f(P)$ where $f(P)$ is a quadratic function of P. How does one fit this model to real data? What qualitative and quantitative predictions can be obtained if one considers replacing the quadratic by a cubic in P? Are there other meaningful extensions of the logistic model? How are the Lotka-Volterra models of competition and predation affected by the assumption that one species grows logistically in the absence of the other?

   For further information, see Michael Olinick.

8. **Topics in Mathematical Bioeconomics**

   Mathematical bioeconomics is concerned with the development, analysis and testing of mathematical models of economic problems relating to the growth and management of biological populations. A typical problem in this field would ask how to maximize the present value of discounted net economic revenue associated with the hunting and capture of whales. How does an optimal strategy vary with the number of competing whaling fleets?

   References: Colin Clark, *Mathematical Bioeconomics*

   For further information, see Michael Olinick.

9. **Browers Fixed Point Theorem**

   In 1912, a Dutch mathematician L E.J. Brouwer proved that every continuous function from a n-cell to itself has at least one fixed point; that is, if $f : B \mapsto B$ is continuous where B is homeomorphic to the set of points at most one unit from the origin in n-space, then there is a point p in B such that $f(p) = p$. This result, a fundamental theorem in topology, has also had many applications in economics and the theory of games. In recent years, new and simpler existence proofs of the fixed point theorem have been discovered. There has also been much progress on the problem of computationally determining fixed points.

   References: Joel Franklin, *Methods of Mathematical Economics.*

   Herbert Scarf, *The Computation of Economic Equilibria.*

   For further information, see Michael Olinick.

10. **Models of Tumor Growth**

    The Gompertz growth law,$dN/dt = bN \ln(K/n)$, is a widely used deterministic model for the growth of tumor. Here N is the number of tumor cells at time t, K is the largest tumor size and b is a positive constant. A thesis in this area would begin with an investigation of the mathematical properties of this model and the statistical tests for deciding when it is a good one. The thesis would then move to a consideration of stochastic models of the tumor growth process.

Reference: B. Hanson and C. Tier, "A Stochastic Model of Tumor Growth", *Mathematical Biosciences*, 59 (1982).

For further information, see Michael Olinick.

11. **Mathematical Models of Conventional Warfare**

    Most defense spending and planning is determined by assessments of the conventional (i.e. non-nuclear) military balance. The dynamic nature of warfare has historically been modeled by a particular simple linked system of differential equations first studied by F. W. Lanchester. The Lanchester approach has recently been challenged and other mathematical models have been proposed.

    Reference: Joshua Epstein, *The Calculus of Conventional War* .

    James Taylor. *Lanchester Models of Warfare.* For further information, see Michael Olinick.

12. **The Fundamental Theorem of Algebra**

    The Fundamental Theorem of Algebra states that every non-constant polynomial with complex coefficients has a complex root. C.F. Gauss was the first person to give a proof of this result; in fact, he discovered four different proofs. All known proofs require some complex analysis. However, the theorem is one of algebra and a purely algebraic proof would be nice to find. Emil Artin has given one that's almost purely algebraic. A senior thesis project could include a presentation of several different types of proof and a search for an algebraic one.

    References: Any text in complex analysis.

    J Munkres, *An Introduction to Topology.*

    Serge Lang, *Algebra* (for Artin's proof).

    For further information, see Priscilla Bremser or David Dorman.

13. **Algebraic Numbers**

    A real number r is *algebraic* if r is the root of a polynomial with integer coefficients. Thus every rational number is algebraic as are many of the more familiar irrational numbers such as the square root of 2 and the l7th root of 3. Cantor proved that the set of algebraic numbers is countable, so that "most" real numbers are not algebraic. Liouville was the first to show explicitly that a certain number was not algebraic. Later in the l9th Century, proofs were discovered that e and π are not algebraic. All these proofs are within the grasp of a senior mathematics major. Although much progress has been made in this century, there are still a number of open questions; for example, is $e + \pi$ algebraic?

    Reference: W. Tabor, In Search of Irrational and Transcendental Numbers, Senior Thesis, 1984.

    For further information Peter Schumer, or David Dorman.

14. **Nonstandard Analysis**

    Would you like to see epsilons and deltas returned to Greek 101, where they belong? Your beginning calculus teachers only pay lip service to them anyway, fudging the definition of limit through phrases like "a tiny bit away" or "as close as you please." Non-standard analysis makes this hand-waving legitimate. In some ways it leaps back in time past the 19th Century godfathers of modern analysis to the founders of calculus by introducing, but in a rigorous way, "infinitesimals" into the real number system. Mathematics is not a static, immutable body of knowledge. New approaches to old problems are constantly being investigated and, if found promising, developed. Nonstandard analysis is a good and exciting example of this mathematical fact of life.

    References: A. Robinson, Non-standard Analysis (1970).

    H Jerome Keisler, *Elementary Calculus* (1976).

    H Jerome Keisler, *Foundations of Infinitesimal Calculus* (1976).

    For further information, see Michael Olinick.

15. **Galois Theory**

    The relation between fields, vector spaces, polynomials, and groups was exploited by Galois to give a beautiful characterization of the automorphisms of fields. From this work came the proof that a general solution for fifth degree polynomial equations does not exist. Along the way it will be possible to touch on other topics such as the impossibility of trisecting an arbitrary angle with straight edge and compass or the proof that the number e is transcendental. This material is accessible to anyone who has had MATH 302.

    Reference: Artin, *Galois Theory* or Herstein, *Topics in Algebra.*

    For additional information, see Priscilla Bremser, David Dorman, or Peter Schumer.

16. **Prime Number Theorem**

    Mathematicians since antiquity have tried to find order in the apparent irregular distribution of prime numbers. Let $\Pi(x)$ be the number of primes not exceeding x. In the 1790's, both Gauss and Legendre proposed that the ratio $\Pi(x)/(x/\ln x)$ would approach 1 as x tended to infinity. Many of the greatest mathematicians of the 19th Century attempted to prove this result and in so doing developed the theory of functions of a complex variable to a very high degree. Partial results were obtained by Chebyshev in 1851 and Riemann in 1859, but the Prime Number Theorem (as it is now called) remained a conjecture until Hadamard and de la Valle' Poussin independently and simultaneously proved it in 1896.

    Reference: A. Koo, "The Prime Number Theorem" Senior Thesis, 1985.

    For further information, see Peter Schumer.

17. **Waring's Problem**

In 1770, the English mathematician Waring stated (without proof) that every positive integer can be expressed as the sum of 4 squares, of 9 cubes, of 19 4th powers, etc. Waring's Problem is to show that for every k there is a finite number $g(k)$ so that all positive integers can be expressed as the sum of $g(k)$ kth powers. In the same year, Lagrange succeeded in demonstrating that $g(2) = 4$, namely that every integer is the sum of 4 squares. The existence of $g(k)$ for several other specific values of k was subsequently proven but it wasn't until 1909 that Hilbert proved the existence of $g(k)$ for every k. However, Hilbert's proof did not determine the numerical value of $g(k)$ for any k.

Senior thesis work in this area could include Lagrange's four square proof, the nine cube problem, Hilbert's inductive proof, and/or related results such as the fact that every sufficiently large integer is expressible as the sum of 7 cubes.

References: Hardy and Wright, *An Introduction to the Theory of Numbers*

Margaret Russell, "Waring's Problem", Senior Thesis, 1985.

Peter Schumer, *Introduction to Number Theory*

For further information, see Peter Schumer.

18. **Twin Primes**

Primes like 3 and 5 or 101 and 103 are called *twin primes* since their difference is only 2. It is unknown whether or not there are infinitely many twin primes. In 1737, Leonard Euler showed that the series $\Sigma 1/p)$ extended over all primes diverges; this gives an analytic proof that there are infinitely many primes. However, in 1919 Viggo Brun proved the following: if q runs through the series of twin primes, then $\Sigma(1/q)$ converges. Hence most primes are not twin primes. Brun's proof is rather difficult but does not depend too heavily on a strong number theory background. A computer search for large twin primes could be fun too.

Reference: E. Landau, *Elementary Number Theory*, Chelsea, 1958; pp. 88-103.

For further information, see Peter Schumer, David Dorman, or Priscilla Bremser.

19. **Continued Fractions**

Have you ever wondered why 22/7 is a pretty good approximation to π or why 355/113 is an excellent one? Do numbers like

$$1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$$

make any sense?

The above are examples of infinite continued fractions (in fact, x is the positive square root of 2). Their properties have been studied over the centuries and many interesting, yet readily accessible, results have been discovered. Moreover, their theory is intimately related to the solution of Diophantine equations, Farey fractions, and the approximation of irrationals by rational numbers.

References: L. Homrighausen, "Continued Fractions", Senior Thesis, 1985.

Rob Jenkins, "On Continued Fractions", Senior Thesis, 1987.

Peter Schumer, *Introduction to Number Theory*

For further information, see Peter Schumer, David Dorman, or Priscilla Bremser.

20. **Additive Bases**

Attempts to prove Goldbach's Conjecture have led to the development of new areas in additive number theory (see Additive Number Theory, topic # 3). One such area originated with the work of the Russian mathematician Schnirelmann. He proved that there is a finite number k so that all integers are the sum of at most k primes. This result may seen innocuous enough, but its proof involved creating the idea of an additive basis and developing the notion of set density, and it involved some interesting combinatorial manipulations. Subsequent work has centered upon results with bases other than primes, determining effective values for k, and studying how sparse a set can be and still generate the integers – the theory of essential components.

Reference: A. Y. Kinchin, *Three Pearls of Number Theory*, Graylock, 1956.

For further information, see Peter Schumer or David Dorman.

21. **Primality Testing and Factoring**

This topic involves simply determining whether a given integer n is prime or composite, and if composite, determining its prime factorization. Checking all trial divisors less than the square root of n suffices but it is clearly totally impractical for large n. Mathematicians have developed very sophisticated methods to deal with this problem. Those methods often utilize the particular form of n – e.g., n may be of the form (a) 2^p1 or (b) $2^{2^q} + 1$. How did Euler determine (a) was prime for $p = 31$ or (b) was composite for $q = 5$? Why did Euler initially think that 1,000,009 was prime before rectifying his mistake? What theorems have been used to allow computers to determine the primality of (a) for$p = 859, 433$?

This could be a very interesting and challenging topic with either a mathematical or computational focus.

Reference: Hans Riesel, *Prime Numbers and Computer Methods for Factorization*

Peter Schumer, *Introduction to Number Theory*

For further information, see Peter Schumer or David Dorman.

22. **Introduction to Analytic Number Theory**

    Analytic number theory involves applying calculus and complex analysis to the study of the integers. Its origins date back to Euler's proof of the infinitude of primes (1737), Dirichlet's proof of infinitely many primes in an arithmetic progression (1837), and Vinogradov's theorem that all sufficiently large odd integers are the sum of three primes (1937). (Did you spot the arithmetic progression in the sentence above?) Other topics could include Chebyshev's theorems on the distribution of primes, the Prime Number Theorem (topic # 16), summatory relationships for various number theoretic functions, Bertrand's postulate, results on lattice points visible from the origin, Kronecker and Weyl's theorems on uniform distributions, and much more.

    There is an endless wealth of topics here for anyone already versed in elementary number theory.

    Reference: T. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

    For further information, see Peter Schumer.

23. **Finite Fields**

    A finite field is, naturally, a field with finitely many elements. For example, $Z/(p)$, where p is a prime number, is a finite field. Are there other types of finite fields? If so, how can their structure be characterized? Are there different ways of representing their elements and operations? A thesis on finite fields could being with these questions and then investigate polynomials and equations over finite fields, applications (in coding theory, for example) and/or the history of the topic.

    For further information, see Priscilla Bremser or David Dorman.

24. **Infinite Products and the Gamma Function**

    "Infinite Products" does for multiplication what Infinite Series does for addition. In what sense can one say that a product of infinitely many factors converges to a number? To what does it converge? Can one generalize the idea of $n!$ for n an integer to $x!$ where x is a real number? This topic is closely related to a beautiful and powerful instrument called the Gamma Function. Infinite products have recently been used to investigate the probability of eventual nuclear war.

    Reference: T.J.I. Bromwich, *An Introduction to the Theory of Infinite Series*.

    For further information, see Michael Olinick.

25. **Computer Simulation and Recognition of Language**

    This thesis would involve some original research into the development of numerical measures that could be used to decide in what natural language (e.g., English, French) a given piece of text was written. We're also interested in investigating whether prose styles of different authors can be distinguished by the computer.

    For further information, see Michael Olinick.

26. **Representation Theory**

    Representation theory is one of the most fruitful and useful areas of mathematics. The theory of group representations was introduced by Frobenius in 1896 as an attempt to generalize the theory of characters of finite abelian groups. The development of the theory was carried on at the turn of the century by Frobenius as well as Shur and Burnside. Since that time the research in representation theory has been extremely active and its applications to all areas of mathematics has been very fruitful. In fact there are some theorems for which only representation theoretic proofs are known. Representation theory also has wide and profound applications outside mathematics. Most notable of these are in chemistry and physics. For example, the theory of molecular bonding, quantum mechanics, and the eight-fold way are best explained in terms of representation theory. A thesis in this area might restrict itself to linear representation of finite groups. Here one only needs background in linear and abstract algebra. Two possible goals are the Frobenius Reciprocity Law and/or Burnside's pq theorem which states that a group whose order is divisible by only two primes is solvable.

    Reference: Serre, *Linear Representations of Finite Groups.*

    For further information, see David Dorman.

27. **Lie Groups**

    Lie groups are all around us. In fact unless you had a very unusual abstract algebra course the ONLY groups you know are Lie groups. (Don't worry there are very important non-Lie groups out there.) Lie group theory has had an enormous influence in all areas of mathematics and has proved to be an indispensable tool in physics and chemistry as well. A thesis in this area would study manifold theory and the theory of matrix groups. The only prerequisites for this topic are calculus, linear and abstract algebra. One goal is the classification of some families of Lie groups.

    Reference: Curtis, *Matrix Groups.*

    For further information, see David Dorman.

28. **Quadratic Forms and Class Numbers**

    The theory of quadratic forms introduced by Lagrange in the late 1700's and was formalized by Gauss in 1801. The ideas included are very simple yet quite profound. A quadratic form is an expression $f(x, y) = ax^2 + bxy + cy^2$ with a and b integers. A typical question one might ask is given a fixed quadratic form, what integers, n , can such a form represent. For example, what integers can be represented by $f(x, y) = x^2 + y^2$? One can show that any prime congruent to 1 modulo 4 can be represented but no prime congruent to 3 modulo 4 can. Of course, 2 can be represented as $f(1, 1)$. A thesis in this area could investigate the classification of definite and indefinite forms, the determination of class numbers and proving the finiteness of class numbers.

Reference: Davenport, *The Higher Arithmetic.*

For further information, see David Dorman.

29. **Generalizations of the Real Numbers**

Let R^n be the vector space of n-tuples of real numbers with the usual vector addition and scalar multiplication. For what values of n can we multiply vectors to get a new element of R^n? The answer depends on what mathematical properties we want the multiplication operation to satisfy. For example, if $n = 1$ we get the ordered field of real numbers with its rich structure which enables us to do most any algebraic or analytic manipulation. If we're just slightly less fussy, for $n = 2$ we get the field of complex numbers. For $n = 4$ we get the division algebra of quaternions. For $n = 8$ we get the weak division algebra called octonians. What happens when $n = 3$?

A thesis in this area would involve learning about the discoveries of these various "composition algebras" and studying the main theorems:

1. The Fundamental Theorem of Algebra
2. Frobenius' Theorem on Division Algebras.

Reference: Simon Altmann, Hamilton, Rodrigues, and the Quaternion Scandal, *Mathematics Magazine*, Vol. 62, No. 5, Dec. 89, 291-308.

For further information, see Peter Schumer.

30. **The Arithmetic-Geometric Inequality and Other Famous Inequalities**

Inequalities are fundamental tools used by many practicing mathematicians on a regular basis. This topic combines ideas of algebra, analysis, geometry, and number theory. We use inequalities to compare two numbers or two functions. Recall the Cauchy-Schwartz inequality and the Triangle inequality from MATH 223. These are examples of the types of relationships that could be investigated in a thesis. You could find different proofs of the inequality, research its history and find generalizations.

References: Hardy, Littlewood, and Plya, *Inequalities*, Cambridge, 1952.

Beckenbach and Bellman, *An Introduction to Inequalities*, MAA, 1961.

For further information, see Bill Peterson.

31. **History of Mathematics**

There are many topics in the history of mathematics that could be developed into a thesis – the history of an individual or group of mathematicians (e.g. Ramanujan or women in mathematics), the history of mathematics in a specific region of the world (e.g. Islamic, Chinese, or the development of mathematics in the U.S.), or the history of a particular branch of mathematics (e.g. calculus, logic, geometry, algebra, or computer science).

For further information, see Michael Olinick or Peter Schumer.

32. **Decision-Theoretic Analysis and Simulation**

Medical researchers and policy makers often face difficult decisions which require them to choose the best among two or more alternatives using whatever data are available. The reformulation of a hypothesis testing problem as a three-decision problem offers a convenient and more suitable way to frame many statistical problems. An axiomatic formulation of a decision problem uses loss functions, various decision criteria such as maximum likelihood and minimax, and Bayesian analysis to lead investigators to good decisions. To compare these approaches to the more traditional Neyman-Pearson Hypotheses testing, computer simulation using massive resampling MATH 311 or its equivalent provide the needed background.

References: Foundations, Concepts and Methods, Springer-Verlag, 1980.

Emerson, J.D. and Tritchler, D. "The Three Decision Problem in Medical Decision Making," *Statistics in Medicine*, 6 (1987), 101-112

For further information, see John Emerson.

33. **Bayesian Statistics: Theory and Decision Making**

The power of modern computers has made possible the analysis of complex data set using Bayesian models and hierarchical models. These models assume that the parameters of a model are themselves random variables and therefore that they have a probability distribution. Bayesian models may begin with prior assumptions about these distributions, and may incorporate data from previous studies, as a starting point for inference based on current data. This project would investigate the conceptual and theoretical underpinnings of this approach, and compare it to the traditional tools of mathematical statistics as studied in MATH 311. It could culminate in an application that uses real data to illustrate the power of the Bayesian approach.

Background for thesis: MATH 310, with MATH 311 a plus.

References: Gelman, A., Carlin, J.B., Stern, H.S., and Rubin, D.B. (1994). *Bayesian Data Analysis*

Lee, Peter M. (1989). *Bayesian Statistics: An Introduction*, Oxford University Press, New York.

Pollard, William E. (1986). *Bayesian Statistics for Evaluation Research: An Introduction*, SAGE

For further information, see John Emerson.

34. **Random Effects Models in Analysis of Variance**

    Measurements which arise from one or more categorical variables that define groups are often analyzed using ANOVA (Analysis of Variance). Linear models specify parameters that account for the differences among the groups. Sometimes these differences exhibit more variability than can be explained by these "fixed effects", and then the parameters are permitted to come from a random distribution, giving "random effects."

    This modeling approach has proved useful and powerful for analyzing multiple data sets that arise from different research teams in different places. For example the "meta-analysis" of data from medical research studies or from social science studies often employs random effects models. This project would investigate random effects models and their applications. A simulation project could illustrate the properties and the strengths of the models when used for making inferences.

    Needed background: MATH 310, with MATH 311 a plus.

    References: Emerson, J.D., Hoaglin, D.C., and Mosteller, F. "A Modified Random-Effects Procedure for Combing Risk Differences in Sets of 2 X 2 Tables from Clinical Trials," *Journal of the Italian Statistical Society* 2 (1993, appeared in January 1995), 269-290.

    Hoaglin, D.C., F. Mosteller, and J. W. Tukey, Eds., *Fundamentals of Exploratory Analysis of Variance*. New York: Wiley, 1991.

    For further information, see John Emerson.

35. **Modern Analysis of Variance**

    Tables containing amounts or measurements often arise as responses to three or more factors, each having two or more levels. Analysis of variance is the study of how the factors, both separately and jointly, account for the variability of the responses. Modern analysis of variance increasingly combines computer-generated ANOVA tables with the use of exploratory and graphical techniques in assessing potentially complex relationships among variables. This investigation would examine models and their assumptions for three-way and higher-way tables. Use of a statistical package like SAS would enable the application of the theory to a data set having current interesting an applications area. Background for thesis: MATH 310, with 311 a plus.

    References: L. Ott, *An Introduction to Statistical Methods and Data Analysis, Duxbury Press*, 1984.

    John D. Emerson, "Transformation and Reformulation in the Analysis of Variance", unpublished draft of a chapter.

    For further information, see John Emerson.

36. **Pseudo-Random Number Generation**

Because a computer is deterministic, it cannot generate truly random numbers. Clever algorithms are currently under investigation for generating a sequence of pseudo-random numbers – a sequence that has most of the statistical properties that make numbers appear random. A thesis project could explore methods of generating pseudo-random numbers from a variety of discrete and continuous probability distributions. The project would culminate in an application to an applied problem that uses statistical simulation. Needed background: MATH 310 and programming experience in C, Pascal or FORTRAN.

References: G. Rothman, "Pseudo-Random Number Generation", Senior Thesis, 1984.

Morgan, B.J.T. *Elements of Simulation*, Chapman and Hall, 1984.

For further information, see John Emerson.

37. **Tilings and Patterns**

Mosaics of multicolored tiles, cobblestone streets, quilts, and honeycombs are examples of tilings. The art of tilings has been studied a great deal, but the science of the designs is a relatively new field of mathematics. Some possible topics in this area are: symmetry, topology, transitivity, tilings using only certain shapes, or tiles with special characteristics, classification of patterns, colorings, and geometry. The problems in this area are easy to state and understand, although not always easy to solve. The pictures are great and the history of tilings and patterns goes back to antiquity. An example of a specific problem that a thesis might investigate is: Devise a scheme for the description and classification of all tilings by angle-regular hexagons.

Reference: Grnbaum and Shephard, *Tilings and Patterns*, Freeman, 1987.

Danzer, Grnbaum, and Shephard, "Equitransitive Tilings, or How to Discover New Mathematics", *Mathematics Magazine*, April, 1987.

For further information, see Priscilla Bremser.

38. **Iterated Function Systems**

Roughly speaking, a *contraction* of the plane is a transformation $f : R^2 \mapsto R^2$ such that if P and Q are any two points in the plane then the distance from $f(P)$ to $f(Q)$ is strictly less than the distance from P to Q – i.e. f decreases all distances. For any contraction f there is a point P_f such that regardless of where P is in the plane, the sequence of points $f(P), f(f(P)), f(f(f(P))), ...$ always converges to P_f!! Now if $F = f_1, f_2, ..., f_n$ is any finite set of contractions then there is not necessarily a single point but nonetheless a compact set C_F such that regardless of where P is in the plane, the sequence of points

$$f_{i1}(P), f_{i2}(f_{i1}(P)), f_{i3}(f_{i2}(f_{i1}(P))), ..$$

(where each of $f_{ij} \in F$) always converges to a point on the set C_F!!

The actual shape that this set C_F depends dramatically on the choice of the set of contractions F. With a little effort C_F can even be made to look like a tree or a flower!!

A thesis in this area would involve learning about these contraction mapping theorems in the plane and in other metric spaces, learning how the choice of contractions effects the shape of C_F and possibly writing computer programs to generate C_F from F. The theoretical work is an extension of the kind of mathematics encountered in MATH 323. Any programming would only require CX213.

Reference: Barnsley, *Fractals Everywhere*

For further information, see Michael Olinick.

39. **Branching Processes**

Consider a population of individuals which produce offspring of the same kind. Associating a probability distribution with the number of offspring an individual will produce in each generation gives rise to a stochastic (i.e., random) model called a branching process. The earliest applications concerned the disappearance of "family names," as passed on from fathers to sons. Modern applications involve inheritance of genetic traits, propagation of jobs in a computer network, and particle decay in nuclear chain reactions. A key tool in the study of branching processes is the theory of generating functions, which is an interesting area of study in its own right.

Reference: Jagers, P. *Branching Processes With Biological Applications.* (New York: Wiley, 1975).

For more information, see Bill Peterson.

40. **The Poisson Process and Order Statistics**

The Poisson Process is a fundamental building block for continuous time probability models. The process counts the number of "events" that occur during the time interval $[0, T]$, where the times between successive events are independent and have a common exponential distribution. Incoming calls to a telephone switchboard, decays of radioactive particles, or student arrivals to the Proctor lunch line are all events that might be modeled in this way. Poisson processes in space (rather than time) have been used to model distributions of stars and galaxies, or positions of mutations along a chromosome. Starting with characterizations of the Poisson process, a thesis might develop some of its important properties and applications. An example: Conditional on n Poisson events having occurred during the interval $[0, T]$, the n occurrence times "look like" (in the sense of joint probability distribution) the ordered values of n points selected uniformly on that interval.

Reference: W. Feller, *An Introduction to Probability Theory and Its Applications*, Volume II, 2e (New York: Wiley, 1971), Chapter 1.

For more information, see Bill Peterson.

41. **Majorization and Schur Convexity**

    Two famous problems in elementary probability are the "Birthday Problem" and the "Coupon Collector's Problem." From the first, we learn that if all 365 possible birth-dates (ignoring leap year) are equally likely, then with 23 people in a room there is a better than even chance that two will share the same birth-date! For the second, imagine that each box of your favorite breakfast cereal contains a coupon bearing one of the letters "P", "R", "I", "Z" and "E". Assuming the letters are equally likely to appear, the expected number of boxes required to collect a set of coupons spelling P-R-I-Z-E is given be 5 * [1 + 1/2 + 1/3 + 1/4 + 1/5] = 11.42

    Now suppose that the "equally likely" assumptions are dropped. Intuition suggests (?) that the chance of a birthday match with 23 people goes up, and the expected time to collect the coupons is longer. But how does one prove such claims? A thesis might investigate the theory of majorization, which provides important tools for establishing these and other inequalities.

    For more information, see Bill Peterson.

42. **Cover Times**

    This is a modern topic combining ideas from probability and graph theory. A "cover time" is the expected time to visit all vertices when a random walk is performed on a connected graph. Here is a simple example (reported by Jay Emerson from his Ph.D. qualifying exam at Yale!): Consider a rook moving on a 2x2 chessboard. From any square on the board, the rook has two available moves. If the successive choices are made by tossing a coin, what is the expected number of moves until the rook has visited each square on the board? In more formal language, this asks for the cover time for the cyclic graph C_4. (Answer: 6 moves).

    References: Blom, G. and Sandell D. (1992), "Cover Times for Random Walks on Graphs." *The Mathematical Scientist*, 17, 111-119.

    For more information, see Bill Peterson

43. **Reliability Theory**

    Reliability theory is concerned with computing the probability that a system, typically consisting of multiple components, will function properly (e.g., the system might be a computer communication network, whose the components are the links and nodes). The components are subject to deterioration and failure effects, which are modeled as random processes, and the status of the system is determined in some way by the status of the components. For example, a series system functions if and only if each component functions, whereas a parallel system functions if and only if at least one component functions. In more complicated systems, it is not easy to express system reliability exactly as a function of component reliabilities, and one seeks instead various bounds on performance.

Reference: S.M. Ross, *Introduction to Probability Models* (Academic Press, 1993), Chapter 9.

For more information, see Bill Peterson.

44. **Measure and Integration**

The Riemann integral studied in MATH 121 and MATH 323 suffers from a few major deficiencies. Specifically, in order to be Riemann integrable, a function must be continuous almost everywhere. However, many interesting functions that show up as limits of integrable functions or even as derivatives do not enjoy this property. Certainly one would want at least every derivative to be integrable. To this end, Henri Lebesgue announced a new integral in 1901 that was completely divorced from the concept of continuity and instead depended on a concept referred to as measure theory. Interesting in their own right, the theorems of measure theory lead to fascinating and paradoxical insight into the structure of sets.

Reference: H.L. Royden, *Real Analysis*

For further information, see Steve Abbott.

45. **Chvátal's Conjecture**

Given a collection $\mathcal{F}$ of sets which is closed downward (a collection in which any subset of a set in $\mathcal{F}$ is also in $\mathcal{F}$) what is the largest intersecting sub-collection? That is, we want a set of sets from $\mathcal{F}$ such that any two sets have a non-empty intersection. What is the structure of such a sub-collection? Chvátal conjectured more than thirty years ago that a largest intersecting sub-collection can always be realized by a star, i.e. a sub-collection of sets in $\mathcal{F}$ each of which contain some fixed element. The conjecture remains open, though some particular cases have been solved.

Reference:

Ian Anderson, *Combinatorics of Finite Sets*, Dover, New York, 1987.

And the homepage of Vasek Chvátal:
http://users.encs.concordia.ca/∼chvatal/conjecture.html

For further information, see John Schmitt.

46. **Combinatorial Nullstellensatz**

Given a finite set of objects and a set of rules placed upon those objects, the first question in combinatorics is whether there exists an arrangement of these objects satisfying the rules. At times this is a trivial question and at other times it can be frustratingly difficult. While having an explicit arrangement of the objects is most desirable, sometimes we, as combinatorists, have to satisfy ourselves with simply knowing such an arrangement exists. Recently, a powerful algebraic technique has been developed by Noga Alon to help answer this existence question. It is known as Combinatorial Nullstellensatz and is a stronger form of Hilbert's Nullstellensatz for particular cases of that statement. This new statement makes it easy for us to know, for example, that any graph with average degree at least 4 and maximum degree at most 5 contains a 3-regular subgraph! It's a beautiful theorem, trust me.

Reference:

Ervin Győri and Vera Sós, *Recent Trends in Combinatorics*, Cambridge, UK, 2001.

For further information, see John Schmitt.

12 How To Give a Good Talk

Joe Gallian has written a short but perceptive article giving very practical suggetions for giving mathematical talks. His article follows.