**MULTIVARIABLE CALCULUS**
**EXAM 3**
**SPRING 2021**

**Name:**
**Honor Code Statement:**

**Directions:** Complete all problems. Justify all answers/solutions. Each problem is worth 10 points; partial credit is given for any progress made. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices; these should not be used under any circumstances. The last two pages contains formulas. Best of luck.

(1) Calculate the given iterated integral and indicate of what region in $\mathbb{R}^3$ that it may be considered to represent the volume.

$$\int_0^5 \int_{-2}^2 (4 - x^2)dx\ dy$$

*Date*: May 28, 2021.

(2) The definition of the double integral can be use to justify that $\iint_D 1 \; dA$ gives the area of $D$. Use this fact to calculate the area of the region bounded by $y = 2x, x = 0$, and $y = 1 - 2x - x^2$. Sketch the region.

(3) Rewrite the given sum of iterated integrals as a single iterated integral by reversing the order of integration and evaluate. Sketch the region.

$$\int_{-2}^{-1}\int_{-\sqrt{y+2}}^{\sqrt{y+2}}(x-y)\ dx\ dy+\int_{-1}^{2}\int_{-\sqrt{y+2}}^{-y}(x-y)\ dx\ dy$$

(4) One may use a triple integral to compute the volume of a ball of radius 1 to be $\frac{4\pi}{3}$. That is, one may show that

$$\iiint_W 1 \, dV = \frac{4\pi}{3},$$

where $W = \{(x, y, z) : |x^2 + y^2 + z^2| \leq 1\}$. Write this triple integral as an iterated integral. Do NOT evaluate it.

(5) Use a change of variables to evaluate the following double integral

$$\iint_D sin(x^2 + y^2)dA,$$

where $D$ is the region in the first quadrant bounded by the coordinate axes and the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. (Note the table of formulas provided at the end of the exam.)

(6) Let $\mathbf{F} = (x^2 + y)\mathbf{i} + (y - x)\mathbf{j}$ and let $\mathbf{x}(t) = (t, t^2), 0 \leq t \leq 1$. Calculate

$$\int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{ds}.$$

(7) Use Green's Theorem to calculate the area under one arch of the cycloid

$$x = (t - sin(t), \ y = (1 - cos(t)).$$

Hint: One first needs to recall the formula that turns the double integral calculation that naturally is given into a line integral. Also, there are two boundary pieces to this region, one is given and the other is (perhaps obviously) the horizontal axis.

**Change of coordinates**

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$

**Change of variables in triple integrals:**

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

# Trigonometric Identities

## Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

## Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

## Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

## Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

## Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.