MULTIVARIABLE CALCULUS
EXAM 3
SPRING 2021

Name:

Honor Code Statement:

70 points
Avg: 62.8
(too easy?!)

Directions: Complete all problems. Justify all answers/solutions. Each problem is worth 10 points; partial credit is given for any progress made. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices; these should not be used under any circumstances. The last two pages contains formulas. Best of luck.

(1) Calculate the given iterated integral and indicate of what region in $\mathbb{R}^3$ that it may be considered to represent the volume.

$$\int_0^5 \int_{-2}^2 (4 - x^2)dx\, dy$$

The volume we are calculating is that of the solid under $z = 4 - x^2$ + above the rectangle $R = \{(x,y) \mid -2 \leq x \leq 2, 0 \leq y \leq 5\}$.

It equals

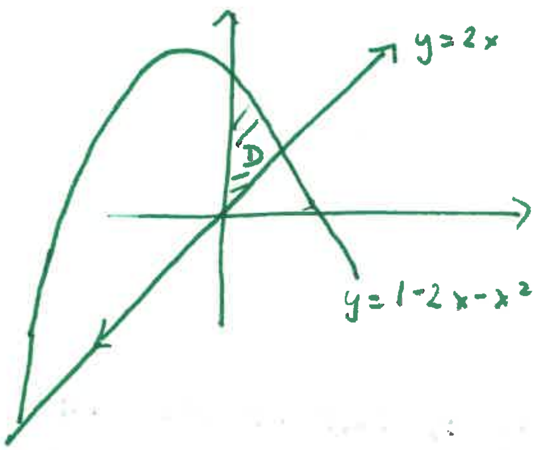
$$\int_0^5 4x - \frac{x^3}{3} \Big|_{-2}^{2} dy = \int_0^5 \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) dy$$

$$= \int_0^5 \frac{32}{3} dy = \frac{32}{3} y \Big|_0^5 = \frac{160}{3}$$

Date: May 28, 2021.

(2) The definition of the double integral can be use to justify that $\iint_D 1\, dA$ gives the area of D. Use this fact to calculate the area of the region bounded by $y = 2x, x = 0$, and $y = 1 - 2x - x^2$. Sketch the region.

This is a naughty problem since it is ambiguous what the region is. We begin with a sketch.



Let's "go" with the choice of region D.

We find the intersection of $y=2x$ and $y = 1-2x-x^2$

$$2x = 1 - 2x - x^2 \iff 0 = -x^2 - 4x + 1$$

$$x = \frac{4 \pm \sqrt{16+4}}{-2}$$

$$x = -2 \pm \sqrt{5}$$

Thus,

$$\iint_D 1\, dA = \int_0^{-2+\sqrt{5}} \int_{2x}^{1-2x-x^2} 1 \, dy\, dx$$

$$= \int_0^{-2+\sqrt{5}} (1 - 4x - x^2)\, dx$$

$$= x - 2x^2 - \frac{x^3}{3} \Big|_0^{\sqrt{5}-2} = \frac{10}{3}\sqrt{5} - \frac{22}{3}$$

(3) Rewrite the given sum of iterated integrals as a single iterated integral by reversing the order of integration and evaluate. Sketch the region.

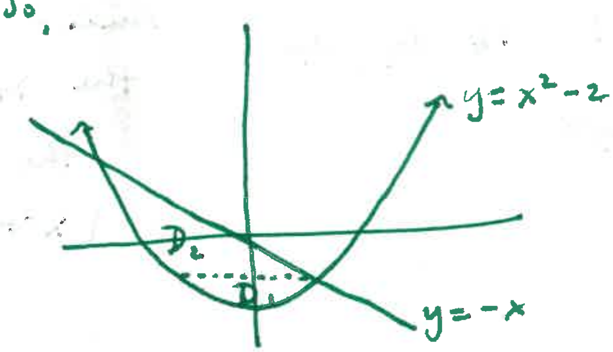
$$\int_{-2}^{-1}\int_{-\sqrt{y+2}}^{\sqrt{y+2}}(x-y)\,dx\,dy+\int_{-1}^{2}\int_{-\sqrt{y+2}}^{-y}(x-y)\,dx\,dy$$

Here's the region:

we have $x=\pm\sqrt{y+2} \iff x^2=y+2 \Rightarrow y=x^2-2$

$x=-y \iff y=-x$.

So,



The integral equals

$$\int_{-2}^{1}\int_{x^2-2}^{-x}(x-y)\,dy\,dx=\int_{-2}^{1} xy-\frac{y^2}{2}\Big|_{x^2-2}^{-x}\,dx$$

$$=\int_{-2}^{1}\left(-x^2-\frac{1}{2}x^2\right)-\left(x^3-2x-\frac{(x^4-4x^2+2)}{2}\right)dx$$

$$=\int_{-2}^{1}-\frac{3}{2}x^2-x^3+2x+\frac{x^4}{2}-2x^2+1\,dx$$

$$=-\frac{x^3}{2}-\frac{x^4}{4}+x^2+\frac{x^5}{10}-\frac{2x^3}{3}+x\Big|_{-2}^{1}=\frac{-9}{20}$$

(4) One may use a triple integral to compute the volume of a ball of radius 1 to be $\frac{4\pi}{3}$. That is, one may show that

$$\iiint_W 1\, dV = \frac{4\pi}{3},$$

where $W = \{(x,y,z) : |x^2+y^2+z^2| \le 1\}$. Write this triple integral as an iterated integral. Do NOT evaluate it.

Possibility 1

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 \, dz\, dy\, dx$$

Note that the shadow of W in the x,y-plane is the ~~circle~~ disk $|x^2+y^2| \le 1$.

Possibility 2

Making a change of coordinates to spherical

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{1} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

(6) Let $\mathbf{F} = (x^2 + y)\mathbf{i} + (y - x)\mathbf{j}$ and let $\mathbf{x}(t) = (t, t^2), 0 \leq t \leq 1$. Calculate

$$\int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{ds}.$$

This vector line integral equals

$$\int_{t_0}^{t_1} \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\,dt$$

$$= \int_0^1 (t^2+t^2, t^2-t) \cdot (1, 2t)\,dt$$

$$= \int_0^1 2t^2 + 2t^3 - 2t^2\,dt$$

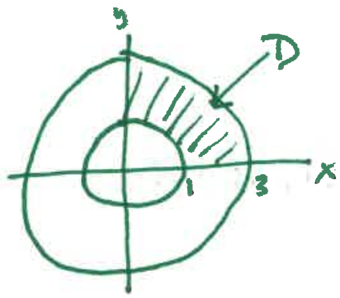
$$= 2\frac{t^4}{4}\Big|_0^1 = \frac{1}{2}$$

(5) Use a change of variables to evaluate the following double integral

$$\iint_D sin(x^2 + y^2)dA,$$

where D is the region in the first quadrant bounded by the coordinate axes and the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. (Note the table of formulas provided at the end of the exam.)

Here's the region



We make a change of variable from Cartesian to polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

Area distortion factor is r.

Thus, the integral becomes

$$\int_0^{\pi/2}\int_1^3 \sin(r^2)\, r\, dr\, d\theta$$

Which by a u-substitution becomes

$$\int_0^{\pi/2} -\frac{1}{2}\cos(r^2)\Big|_1^3 \, d\theta = \int_0^{\pi/2} -\frac{1}{2}(\cos 9 - \cos 1)\, d\theta$$

$$= \theta\left(-\frac{1}{2}(\cos 9 - \cos 1)\right)\Big|_0^{\pi/2} = \frac{\pi}{4}(\cos 1 - \cos 9).$$

(7) Use Green's Theorem to calculate the area under one arch of the cycloid

$$x = (t - sin(t), \ y = (1 - cos(t)).$$

Hint: One first needs to recall the formula that turns the double integral calculation that naturally is given into a line integral. Also, there are two boundary pieces to this region, one is given and the other is (perhaps obviously) the horizontal axis.

We first sketch the arch (and the corresponding region).

For

Note the portion of the horizontal axis can be parametrized as

$$\vec{x}(t) = (2\pi - 2\pi t, 0)$$

for $0 \le t \le 1$.

Recall from page 430 of the text that

$$\text{area of } D = \frac{1}{2} \oint_C -y\,dx + x\,dy = \frac{1}{2} \oint_{C_1} -y\,dx + x\,dy$$

$$+ \frac{1}{2} \int_{C_2} -y\,dx + x\,dy$$

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W*} f(x(u,v,w), y(u,v,w), z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.