MULTIVARIABLE CALCULUS
EXAM 3
SPRING 2018

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices. The last two pages contains formulas. Best of luck.

(1) [10 points] Compute the following iterated integral:

$$\int_{-5}^{5}\int_{-1}^{2}(5-|y|)dx\ dy$$

$$= \int_{-5}^{5} (5-|y|)x \Big|_{-1}^{2} dy$$

$$= \int_{-5}^{5} 3(5-|y|)\, dy$$

Q: why did people struggle with $\int |y|\, dy$?

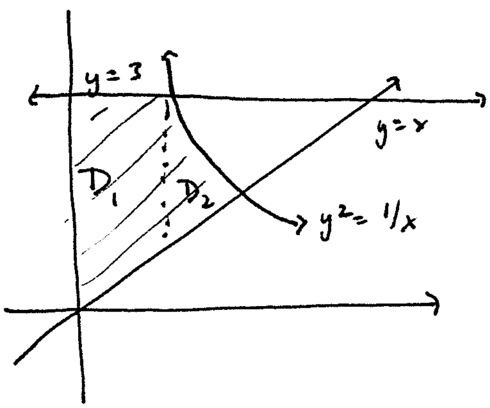
$$= \int_{-5}^{5} 15 - 3|y|\, dy = \int_{-5}^{5} 15\, dy + \int_{-5}^{0} 3y\, dy - \int_{0}^{5} 3y\, dy$$

$$= 15y \Big|_{-5}^{5} + \frac{3}{2}y^2 \Big|_{-5}^{0} - \frac{3}{2}y^2 \Big|_{0}^{5}$$

$$= 150 - \frac{75}{2} - \frac{75}{2} = 75$$

Date: May 18, 2018.

(2) [10 points] Evaluate $\iint_D 3y\, dA$, where D is the region bounded by $xy^2 = 1, y = x, x = 0$ and $y = 3$. BEFORE proceeding, please make a sketch of the region! Graph paper is available.



$$\iint_D 3y\, dA =$$

$$\int_0^{1/9}\int_x^3 3y\, dy\, dx + \int_{1/9}^1 \int_x^{1/\sqrt{x}} 3y\, dy\, dx =$$

$$\int_0^{1/9} \frac{3y^2}{2}\Big|_x^3 dx + \int_{1/9}^1 \frac{3}{2}y^2 \Big|_x^{1/\sqrt{x}} dx =$$

$$\int_0^{1/9} \frac{27}{2} - \frac{3}{2}x^2\, dx + \int_{1/9}^1 \frac{3}{2}\cdot\frac{1}{x} - \frac{3}{2}x^2 dx =$$

$$\left(\frac{27}{2}x - \frac{1}{2}x^3\right)\Big|_0^{1/9} + \left(3/2 \ln x - \frac{1}{2}x^3\right)\Big|_{1/9}^1 =$$

$$\frac{3}{2} - \frac{1}{2}\left(\frac{1}{9}\right)^3 + \left(\frac{3}{2}\ln(1) - \frac{1}{2}\right) - \left(\frac{3}{2}\ln\frac{1}{9} - \frac{1}{2}\left(\frac{1}{9}\right)^3\right) =$$

$$1 - \frac{3}{2}\ln 1/9 =$$

$$1 + \ln(1/9)^{-3/2} = 1 + \ln 27 = 1 + 3\ln 3$$

(3) [5 points] In the previous question you evaluated an iterated integral. Write an equivalent iterated integral with the order of integration reversed. (You need not evaluate this new integral.)

$$\iint_D 3y\, dA = \int_0^1 \int_0^y 3y\, dx\, dy + \int_1^3 \int_0^{1/y^2} 3y\, dx\, dy$$

(4) [5 points] Each of the following statements is false. Correct the statement (in as minimal a way as possible) and without simply negating the solution.

- If $\mathbf{x}$ and $\mathbf{y}$ are two one-one parametrizations of the same curve and $\mathbf{F}$ is a continuous vector field, then $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$.

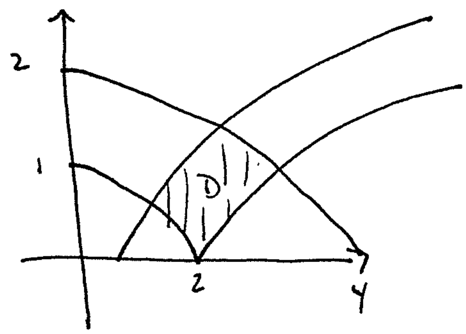
  Insert "orientation preserving".

- Suppose that $f(x) > 0$ for all x. Let $\mathbf{F} = f(x)\mathbf{i}$. If C is the vertical line segment from $(0,0)$ to $(0,3)$, then $\int_C \mathbf{F} \cdot d\mathbf{s} > 0$.

  One possibility (of several): $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(5) [10 points] Evaluate the following integral where D is the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = 1, x^2 - y^2 = 4$ and the ellipses $x^2/4 + y^2 = 1, x^2/16 + y^2/4 = 1$. Graph paper is attached to help you sketch this region. To do this integral, I recommend using the change of variables $u = x^2 - y^2$ and $v = x^2/4 + y^2$. (Hint: Don't work too hard to find the limits of integration, stare at the equations you have for a bit.)

$$\iint_D \frac{xy}{y^2 - x^2} dA$$



Using the change of variables, we first find the area distortion factor.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ x/2 & 2y \end{vmatrix} = 4x - (-xy) = 5xy$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5xy}$$

Thus,

$$\iint_D \frac{xy}{y^2 - x^2} dA = \int_1^4 \int_1^4 \frac{xy}{(-u)} \cdot \frac{1}{5xy} \, du\,dv = \frac{-1}{5} \int_1^4 \int_1^4 \frac{1}{u} \, du\,dv$$

$$= \frac{-1}{5} \int_1^4 \ln(u) \Big|_1^4 dv = \frac{-1}{5} \int_1^4 \ln 4 \, dv$$

$$= \frac{-1}{5} v \ln 4 \Big|_1^4 = \frac{-3}{5} \ln 4$$

(6) [10 points] Calculate the following scalar line integral where $\mathbf{x}(t) = (\cos 4t, \sin 4t, 3t)$ and $0 \leq t \leq 2\pi$.

$$\int_{\mathbf{x}} 3x + xy + z^3 ds$$

First we find $\vec{x}'(t) = (-4\sin 4t, 4\cos 4t, 3)$

and so $\|\vec{x}'(t)\| = \sqrt{16\sin^2 4t + 16\cos^2 4t + 9}$

$= \sqrt{25} = 5$

So by definition of the scalar line integral, we obtain

$$\int_{\vec{x}} 3x + xy + z^3 ds = 5\int_0^{2\pi} 3\cos 4t + \cos 4t \sin 4t + 27t^3 \, dt$$

$$= 5\left(\frac{3}{4}\sin 4t + \frac{1}{8}\sin^2 4t + 27\frac{t^4}{4}\right)\Big|_0^{2\pi}$$

$$= 5\left(\frac{27}{4}(2\pi)^4\right) = 540\,\pi^4$$

(7) [10 points] Use Green's theorem to find the area enclosed by the hypocycloid $\mathbf{x}(t) = (\cos^3(t), \sin^3(t))$, for $0 \leq t \leq 2\pi$. Hint: the area is given by $\iint_D dy\, dx$.

Recall that area is give by $\iint_D dy\, dx$, which by Green's Theorem is equal to

$$\frac{1}{2}\oint_{\partial D} -y\, dx + x\, dy = \frac{1}{2}\int_0^{2\pi} -\sin^3 t \cdot 3\cos^2 t\,(-\sin t) + \cos^3 t \cdot 3\sin^2 t \cos t\, d$$

$$= \frac{1}{2}\int_0^{2\pi} 3\sin^4 t \cos^2 t + 3\cos^4 t \sin^2 t\, dt$$

$$= \frac{3}{2}\int_0^{2\pi} \sin^2 t \cos^2 t\,(\sin^2 t + \cos^2 t)\, dt$$

$$= \frac{3}{2}\int_0^{2\pi} \sin^2 t \cos^2 t\, dt = \frac{3}{2}\int \sin^2 t\,(1-\sin^2 t)\, dt$$

$$= \frac{3}{2}\int_0^{2\pi} \sin^2 t - \sin^4 t\, dt = \frac{3}{2}\int_0^{2\pi} \frac{1-\cos 2t}{2} - \left(\frac{1-\cos 2t}{2}\right)^2 dt$$

$$= \frac{3}{2}\int_0^{2\pi} 1 - \frac{\cos 2t}{2} - \left(\frac{1 - 2\cos 2t + \cos^2 2t}{4}\right) dt$$

$$= \frac{3}{2}\int_0^{2\pi} \frac{1}{4} - \frac{\cos^2 2t}{4}\, dt = \frac{3}{2}\int_0^{2\pi} \frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{3}{2}\int_0^{2\pi} \frac{1}{8} - \frac{1}{8}\cos 2t\, dt = \frac{3}{2}\left(\frac{1}{8}t - \frac{\sin 2t}{16}\right)\Big|_0^{2\pi}$$

$$= \frac{3}{16} \cdot 2\pi = \frac{3}{8}\pi$$

Phew!

(8) [5 points] Suppose that the temperature at a point in the cube

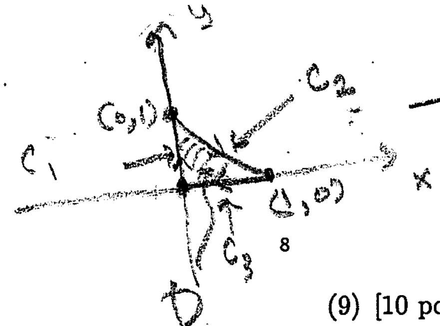
$$W = [-1,1] \times [-1,1] \times [-1,1]$$

varies in proportion (with constant of proportionality k) to the square of the point's distance from the origin. Set up, but do not evaluate, an integral that gives the average temperature of the cube.

By given: $T(x, y, z) = k(x^2 + y^2 + z^2)$

$$\text{Average Temperature} = \frac{\iiint_W T \, dV}{\text{Volume of } W}$$

$$= \frac{k \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} x^2 + y^2 + z^2 \, dx \, dy \, dz}{8}$$



This orientation has region D to the right of the curve C.

(9) [10 points] Use Green's theorem to find the work done by the vector field $\mathbf{F} = x^2y\mathbf{i} + (x+y)y\mathbf{j}$ in moving a particle from the origin along the y-axis to the point $(0,1)$, then along the line segment from $(0,1)$ to $(1,0)$, and then from $(1,0)$ back to the origin along the x-axis.

This is such a nice student solution – I had to have it!

To find work done by a vector field on a path we use a vector line integral. In this case our path is C so we seek $\oint_C F\,ds$. We can partition C into 3 segments and parametrize each as follows:
$C_1 = \{(0,t) \mid 0 \le t \le 1\}$, $C_2 = \{(t, 1-t) \mid 0 \le t \le 1\}$,
$C_3 = \{(1-t, 0) \mid 0 \le t \le 1\}$. Thus the line integral can be rewritten as $\int_0^1 (0, t^2)\cdot(0,1)\,dt + \int_0^1 (t^2 - t^3, t)\cdot(1,-1)\,dt$
$= \int_0^1 t^2\,dt + \int_0^1 (-t^3 + t^2 - t)\,dt = \frac{1}{3}t^3\Big|_0^1 + \left(-\frac{1}{4}t^4 + \frac{1}{3}t^3 - \frac{1}{2}t^2\right)\Big|_0^1 =$
$\frac{1}{3} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2} = \frac{2}{3} - \frac{3}{4} = \boxed{-\frac{1}{12}}$. ✓ This however was found NOT using Green's theorem! I will do that Now! By Green's theorem $\oint_{C^*} M\,dx + N\,dy = \iint_D (N_x - M_y)\,dA$ where C^* is the opposite orientation of C so D is to the left. Now we calculate $N_x = y$ and $M_y = x^2$
Evaluating the integral, By Theorem 2: this double integral becomes the iterated integral: $\int_0^1\int_0^{1-x} (y - x^2)\,dy\,dx = \int_0^1 \left(\frac{1}{2}y^2 - x^2y\right)\Big|_0^{1-x}$
$= \int_0^1 \left(\frac{1}{2}(1-x)^2 - x^2 + x^3\right)dx = \int_0^1 \left(\frac{1}{2}x^2 - x + \frac{1}{2} - x^2 + x^3\right)dx = \int_0^1 \left(-\frac{1}{2}x^2 - x + \frac{1}{2} + x^3\right)dx$
$= \left(-\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}x^4\right)\Big|_0^1 = -\frac{1}{6} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} + \frac{1}{4} = \frac{-2}{12} + \frac{3}{12} = \frac{1}{12}$. We must Now negate this value as C^* had reversed orientation of C so the work done by the vector field is $-\frac{1}{12}$

perfecto!

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W*} f(x(u,v,w), y(u,v,w), z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.