MULTIVARIABLE CALCULUS
EXAM 3
FALL 2019

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices; these should not be used under any circumstances. The last two pages contains formulas. Best of luck.

(1) [10 points] Let D be the region in $\mathbb{R}^2$ bounded by $xy^2 = 1, y = x, x = 0$, and $y = 3$. Sketch the region. Evaluate the following double integral,

$$\iint_D 3y \, dA.$$

The properties that the integrand has to allow a change from a double integral to an iterated integral are ______________________________ (fill-in-the-blank).

Date: December 13, 2019.

(2) [10 points] Rewrite the given sum of iterated integrals as a single iterated integral by reversing the order of integration. Sketch the region of integration. Evaluate the new integral obtained.

$$\int_0^1 \int_0^x \sin x \; dy \; dx + \int_1^2 \int_0^{2-x} \sin x \; dy \; dx$$

(3) [5 points] If W is an elementary region in $\mathbb{R}^3$ and we wished to compute the volume of W using a triple integral $\iiint_W f\ dV$, what should the integrand f be as a function to make this computation? Why?

(4) [10 points] Change the order of integration of

$$\int_0^1 \int_0^1 \int_0^{x^2} f(x,y,z)\ dz\ dy\ dx$$

to give five other equivalent iterated integrals.

(5) [10 points] We will find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

The plane intersects the paraboloid in the circle with equation ______________________________. Thus the solid lies under the paraboloid and above a circular disk D. In polar coordinates, D is given by ____

The volume is given by $V = \iint_D (1 - x^2 - y^2)\ dA$. This would not be easy to evaluate in cartesian coordinates since the limits of integration would involve ______________________________. So we will make a change of variables from cartesian to polar coordinates. Compute this volume by making this change.

(6) [10 points] Find the work done by the force field $\mathbf{F} = x^2y\mathbf{i} + z\mathbf{j} + (2x - y)\mathbf{k}$ on a particle as the particle moves along a straight line from $(1, 1, 1)$ to $(2, -3, 3)$.

(7) [5 points] State the hypothesis of Green's Theorem. I've stated the conclusion.

Then

$$\oint_C M\ dx + N\ dy = \iint_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\ dx\ dy.$$

(8) [10 points]. The area of a region in the plane can be computed as $\iint_D 1\, dx\, dy$. Green's Theorem allows us to compute this double integral as the following vector line integral

$$\oint_C -\frac{1}{2}y\ dx + \frac{1}{2}x\ dy$$

by setting $M(x,y) = -\frac{1}{2}y$ and $N(x,y) = \frac{1}{2}x$.

Another possibility for computing this double integral as a vector line integral is

$$\oint_C x\ dy$$

by setting $M(x,y) = 0$ and $N(x,y)$ equal to what?

What's a third possibility for the computing this double integral as a vector line integral and what are the choices for $M(x,y)$ and $N(x,y)$ that yield this?

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2 + y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x, y, z)dxdydz = \int\int\int_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w))|\frac{\partial(x, y, z)}{\partial(u, v, w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x, y, z)}{\partial(u, v, w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.