MULTIVARIABLE CALCULUS
EXAM 3
FALL 2019

Name: Solution Key

Honor Code Statement: I have neither given nor received unauthorized aid on this exam

Directions: Complete all problems. Justify all answers/solutions. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices; these should not be used under any circumstances. The last two pages contains formulas. Best of luck.

(1) [10 points] Let D be the region in $\mathbb{R}^2$ bounded by $xy^2 = 1, y = x, x = 0$, and $y = 3$. Sketch the region. Evaluate the following double integral,

$$\iint_D 3y \, dA.$$

The properties that the integrand has to allow a change from a double integral to an iterated integral are bounded and continuous (as Fubini requires). (fill-in-the-blank).

See the attached sketch of D.

We must partition D into two regions. This allows us to write the double integral as

$$\int_0^{1/9}\int_x^3 3y \, dy\, dx + \int_{1/9}^1 \int_x^{1/\sqrt{x}} 3y \, dy\, dx$$

$$= \int_0^{1/9} \frac{3y^2}{2}\Big|_x^3 dx + \int_{1/9}^1 \frac{3}{2}y^2 \Big|_x^{1/\sqrt{x}} dx$$

$$= \int_0^{1/9}\left(\frac{27}{2} - \frac{3x^2}{2}\right) dx + \int_{1/9}^1 \frac{3}{2}\cdot\frac{1}{x} - \frac{3}{2}x^2 = \left(\frac{27}{2}x - \frac{x^3}{2}\right)\Big|_0^{1/9} + \left(\frac{3}{2}\ln x - \frac{x^3}{2}\right)\Big|_{1/9}^1$$

$$= \left(\frac{27}{2}\cdot\frac{1}{9} - \frac{\left(\frac{1}{9}\right)^3}{2}\right) - (0) + \left[\left(0 - \frac{1}{2}\right) - \left(\frac{3}{2}\ln\frac{1}{9} - \frac{(1/9)^3}{2}\right)\right]$$

$$= \frac{3}{2} - \frac{1}{2} - \frac{3}{2}\ln(1/9) = 1 + 3\ln 3$$

Date: December 13, 2019.

(2) [10 points] Rewrite the given sum of iterated integrals as a single iterated integral by reversing the order of integration. Sketch the region of integration. Evaluate the new integral obtained.

$$\int_0^1 \int_0^x \sin x \; dy \; dx + \int_1^2 \int_0^{2-x} \sin x \; dy \; dx$$

See the attached sketch.

We seek to integrate w.r.t. x first, then y.

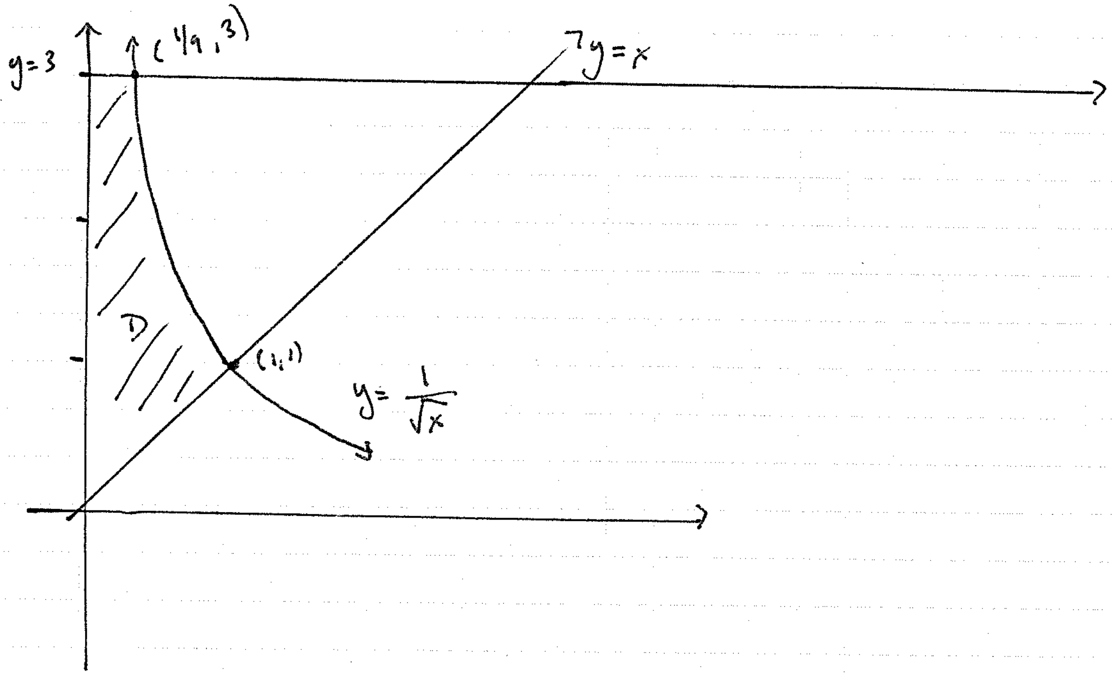
$$\int_0^1 \int_y^{2-y} \sin x \quad dx \quad dy = \int_0^1 -\cos x \Big|_y^{2-y} \; dy$$

$$= \int_0^1 -\cos(2-y) + \cos y \;\; dy$$

$$= \sin(2-y) + \sin y \Big|_0^1 = (\sin 1 + \sin 1) - (\sin 2 + \sin 0)$$

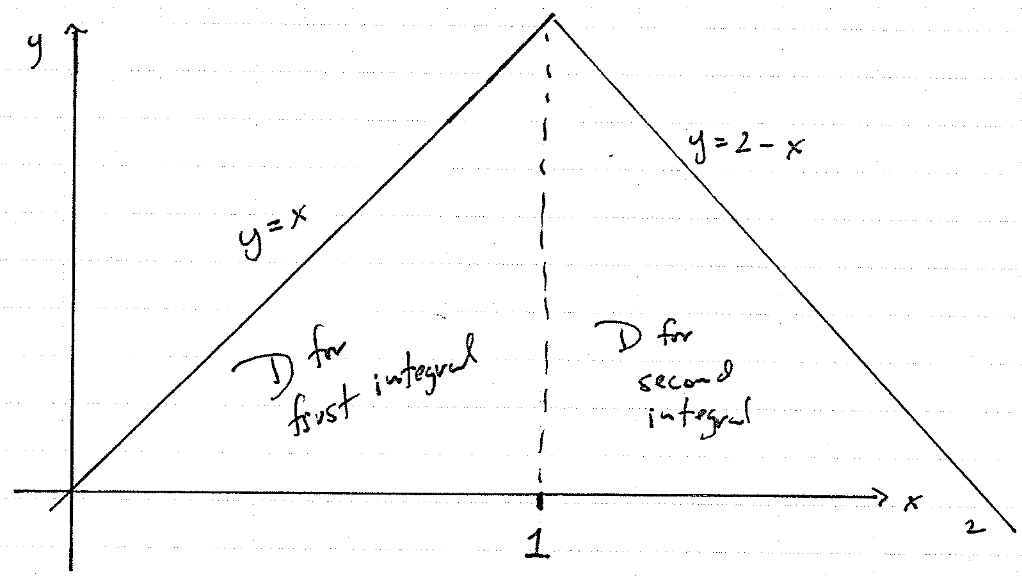
$$= 2\sin 1 - \sin 2$$

①



y=3
(1/9, 3)
y = x
D
(1,1)
y = 1/√x

②



y
y = 2 - x
y = x
D for first integral
D for second integral
x
1
2

(3) [5 points] If W is an elementary region in $\mathbb{R}^3$ and we wished to compute the volume of W using a triple integral $\iiint_W f\, dV$, what should the integrand f be to make this computation? Why?

The integrand should be $f = 1$. This follows from the definition of the triple integral as the limit of a Riemann Sum.

(4) [10 points] Change the order of integration of

$$\int_0^1 \int_0^1 \int_0^{x^2} f(x,y,z)\; dz\; dy\; dx$$

to give five other equivalent iterated integrals.

$z=0$ is the x,y-plane ; $z = x^2$ is a parabolic trough.

$y=0$ is the x,z-plane ; $y=1$ is a plane parallel to the x,z-plane

$x=0$ is the y,z-plane ; $x=1$ is a plane parallel to the y,z-plane.

The region W is in the first/positive octant.

The other 5 forms are:

"z-first" "y first" "x first"

$$\int_0^1\int_0^1\int_0^{x^2} f\; dz\, dx\, dy = \int_0^1\int_0^{x^2}\int_0^1 f\; dy\, dz\, dx = \int_0^1\int_0^1\int_{\sqrt{z}}^1 f\; dx\, dz\, dy$$

$$= \int_0^1\int_{\sqrt{z}}^1\int_0^1 f\; dy\, dx\, dz = \int_0^1\int_0^1\int_{\sqrt{z}}^1 f\; dx\, dy\, dz$$

(5) [10 points] We will find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

The plane intersects the paraboloid in the circle with equation $x^2+y^2=1$. Thus the solid lies under the paraboloid and above a circular disk D. In polar coordinates, D is given by $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

The volume is given by $V = \iint_D (1 - x^2 - y^2)\, dA$. This would not be easy to evaluate in cartesian coordinates since the limits of integration would involve radicals. So we will make a change of variables from cartesian to polar coordinates. Compute this volume by making this change.

We have $r^2 = x^2 - y^2$ and a "distortion factor" of r.

$$\iint_D (1-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{1}{4}\theta\Big|_0^{2\pi}$$

$$= \frac{2\pi}{4} = \frac{\pi}{2}$$

(6) [10 points] Find the work done by the force field $\mathbf{F} = x^2y\mathbf{i} + z\mathbf{j} + (2x - y)\mathbf{k}$ on a particle as the particle moves along a straight line from $(1, 1, 1)$ to $(2, -3, 3)$.

This straight line path may be written as

$$\vec{x}(t) = (t+1,\ 1-4t,\ 1+2t),\quad 0 \le t \le 1.$$

Thus, $\vec{x}\,'(t) = (1, -4, 2)$.

$$\text{Work} = \int_C \vec{F} \cdot d\vec{s} = \int_0^1 (1+t)^2(1-4t)\cdot 1 - 4(1+2t) + 2\big(2(t+1) - (1-4t)\big)dt$$

$$= \int_0^1 (-4t^3 - 7t^2 + 2t - 1)\Big|_0^1 = \frac{-10}{3}$$

(7) [5 points] State the hypothesis of Green's Theorem. I've stated the conclusion.

Let D be a closed bounded region in $\mathbb{R}^2$ with boundary $C = \partial D$ consisting of finitely many simple, closed, piecewise C^1 curves. Orient the curves so that D is on the left as one traverses C. Let $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ be a vector field of class C^1 throughout D.

Then

$$\oint_C M\ dx + N\ dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy.$$

(8) [10 points]. The area of a region in the plane can be computed as $\iint_D 1\, dx\, dy$. Green's Theorem allows us to compute this double integral as the following vector line integral

$$\oint_C -\frac{1}{2}y\, dx + \frac{1}{2}x\, dy$$

by setting $M(x,y) = -\frac{1}{2}y$ and $N(x,y) = \frac{1}{2}x$.

Another possibility for computing this double integral as a vector line integral is

$$\oint_C x\, dy$$

by setting $M(x,y) = 0$ and $N(x,y)$ equal to what?

What's a third possibility for the computing this double integral as a vector line integral and what are the choices for $M(x,y)$ and $N(x,y)$ that yield this?

For the first question, I've told you the answer $N(x,y) = x$. Note that in this way $\frac{\partial N}{\partial x} = 1$ and $\frac{\partial M}{\partial y} = 0$ so that $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{y}\right) = 1$, the integrand we seek to have.

We could also have chosen $M(x,y) = -y$ and $N(x,y) = 0$. In this way we have

$\frac{\partial N}{\partial x} = 0$ and $\frac{\partial M}{\partial y} = -1$ so that $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 0 - (-1) = 1$

the integrand we seek to know. The corresponding vector line integral is thus

$$\oint -y\, dx$$

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.