MULTIVARIABLE CALCULUS
EXAM 3
FALL 2018

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid.

65 points

average 55 points

Directions: Complete all problems. Justify all answers/solutions. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices; these should not be used under any circumstances. The last two pages contains formulas. Best of luck.

(1) [10 points] Compute the following iterated integral:

$$V = \int_{-2}^{3}\int_{0}^{1} (|x|\sin(\pi y))\, dy\, dx$$

$$V = \int_{-2}^{3} |x|\left(\frac{-1}{\pi}\right)\cos(\pi y)\Big|_0^1\, dx \qquad \text{as } \int \sin u\, du = -\cos u + C$$

$$= \int_{-2}^{3} -\frac{|x|}{\pi}\cos\pi - \left(-\frac{|x|}{\pi}\cos 0\right) dx \qquad \text{by evaluating at upper and lower limits of integration}$$

$$= \int_{-2}^{3} \frac{2|x|}{\pi}\, dx$$

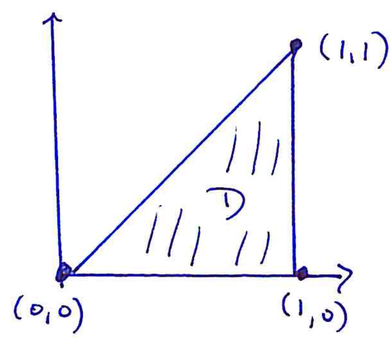
$$= \int_{-2}^{0} \frac{-2x}{\pi}\, dx + \int_{0}^{3} \frac{2x}{\pi}\, dx \qquad \text{as } |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$= \frac{-2x^2}{2\pi}\Big|_{-2}^{0} + \frac{2x^2}{2\pi}\Big|_0^3 = \left(0 + \frac{8}{2\pi}\right) + \left(\frac{18}{2\pi} + 0\right) = \frac{26}{2\pi} = \frac{13}{\pi}$$

Date: December 11, 2018.

(2) [10 points] Evaluate $\iint\limits_D e^{x^2}\, dA$, where D is the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Identify whether D is a Type I, Type II or Type III elementary region.

We begin by sketching the region D.



D may be described as

$$D = \{ (x,y) : 0 \le x \le 1 ,\ 0 \le y \le x \}$$

or

$$D = \{ (x,y) : 0 \le y \le 1 ,\ y \le x \le 1 \}$$

Thus D is of Type III.

Due to Fubini's Theorem We may rewrite the double integral as an iterated integral in two ways:

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy \qquad \text{or} \qquad \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

The first way proves impossible for us as we don't have an integrand for e^{x^2}. Thus, we pursue the second:

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 e^{x^2} \cdot y \Big|_0^x dx = \int_0^1 xe^{x^2} - 0\, dx$$

$= \int_0^1 x e^{x^2} dx$. This we may integrate via a u-substitution: $u = x^2$ $du = 2x\, dx$.

We obtain $\frac{1}{2} e^{x^2} \Big|_0^1 = \frac{1}{2} e^1 - \frac{1}{2} e^0 = \frac{e}{2} - \frac{1}{2}$.

(3) [5 points] Let f be a continuous function on an elementary region D of the plane. Define the *extension of* f. In two sentences, what purpose does this definition serve?

$$f^{ext}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

The ~~pury~~ purpose is so that we may define the double integral of f over D as the double integral of f^{ext} over a rectangle R that encloses D.

(4) [5 points] Calculate the scalar line integral of the function $f(x, y, z) = x^2 z$ over the path $\mathbf{x}(t) = (3t, 2t, 3t), 0 \le t \le 1$.

The scalar line integral of f along the C^1 path $\vec{x}$ is $\int_a^b f(x(t)) \|\vec{x}\,'(t)\| dt$.

So we compute $\int_0^1 (3t)^2(3t)\sqrt{3^2+2^2+3^2}\, dt$

$$= \int_0^1 27\sqrt{22}\, t^3\, dt = \frac{27\sqrt{22}\, t^4}{4}\Big|_0^1$$

$$= \frac{27\sqrt{22}}{4}$$

(5) [10 points] Write the following triple integral

$$\iiint\limits_W z\, dV,$$

where W is the region bounded by $z = 0$, $x^2 + 4y^2 = 4$ and $z = x + 2$, as an iterated integral with order of integration z, then y, then x. You need not compute the integral. A sketch of the region W as given by Maple is provided. What theorem allows us to do this? and what properties does the integrand z have to justify its application?

$$\iiint\limits_W z \, dV = \int_{-2}^{2} \int_{y=-\sqrt{1-\frac{x^2}{4}}}^{y=\sqrt{1-\frac{x^2}{4}}} \int_{z=0}^{z=x+2} z \, dz\, dy\, dx$$

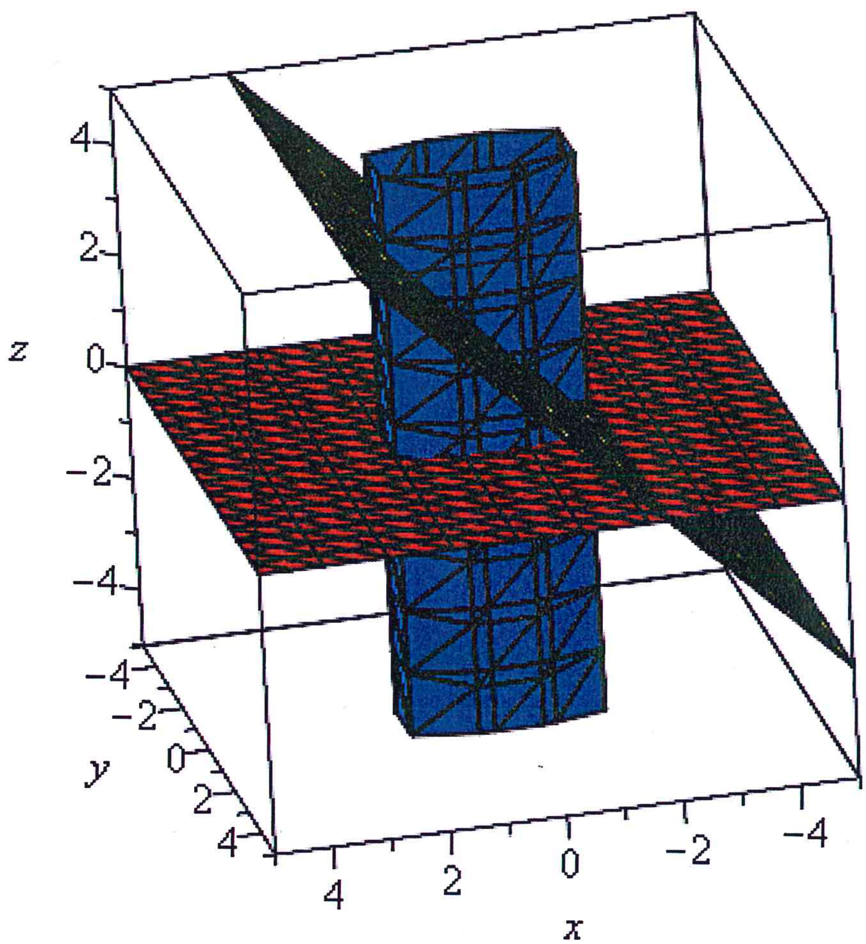
Fubini's Theorem allows us to turn triple integrals into iterated integrals.

The integrand z is continuous - it is a polynomial and so continuous everywhere, including on W. It is bounded on W.

```
> with(plots);
```

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot] **(1)**

> implicitplot3d$([z = 0, x^2 + 4 \cdot y^2 = 4, z = x + 2], x = -5..5, y = -5..5, z = -5..5, color = [red, blue, yellow])$;



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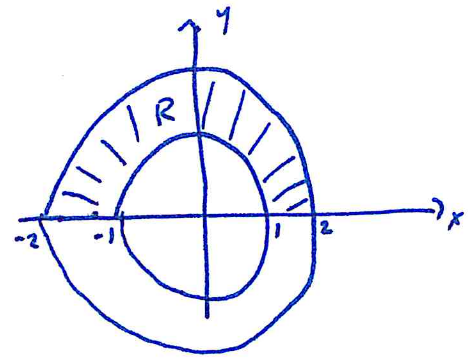
(6) [10 points] (from Stewart) Evaluate

$$\iint\limits_R (3x + 4y^2)\, dA,$$

where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

We begin with a sketch of R.



The circular nature of this region suggests that a change of variables is called for: $x = r\cos\theta$, $y = r\sin\theta$

Recall that with this change of variables, the Jacobian is r.

The polar coordinates used to describe R are $1 \le r \le 2$ and $0 \le \theta \le \pi$.

The integral becomes

$$\int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r \; dr\, d\theta$$

distributing r

$$= \int_0^\pi \int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta \; dr\, d\theta$$

integrating w.r.t. r

$$= \int_0^\pi \frac{3r^3}{3}\cos\theta + \frac{4r^4}{4}\sin^2\theta \Big|_{r=1}^{2} d\theta$$

evaluating

$$= \int_0^\pi 7\cos\theta + 15\sin^2\theta \; d\theta$$

using a trig identity

$$= \int_0^\pi 7\cos\theta + 15\,\frac{(1-\cos 2\theta)}{2}\, d\theta$$

integrating w.r.t. θ

$$= 7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta \Big|_0^\pi = \left(0 + \frac{15}{2}\pi - 0\right) - (0 + 0 - 0) = \frac{15}{2}\pi$$

[5 points]

(7) Suppose that C is the curve $y = f(x)$, oriented from $(a, f(a))$ to $(b, f(b))$ where $a < b$ and where f is positive and continuous on $[a, b]$. If $\mathbf{F} = y\mathbf{i}$, show that the value of

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

is the area under the graph of f between $x = a$ and $x = b$.

We may parametrize the curve C as $\vec{x}(t) = (t, f(t))$, $a \leq t \leq b$. Then we have $\vec{x}\,'(t) = (1, f'(t))$.

Recall that $\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt$.

Thus this vector line integral is

$$\int_a^b (f(t), 0) \cdot (1, f'(t))\, dt = \int_a^b f(t)\, dt$$

which we know to be the area under the graph of f from a to b.

N.B. There's also an approach to this problem using Green's Theorem.

(8) [10 points] Let f and g be single-variable functions of class C^1. Let C be a piecewise C^1, simple, closed curve in $\mathbb{R}^2$. Compute

$$\oint_C f(x)dx + g(y)dy.$$

Justify your solution.

We recognize this integral as the differential form of the line integral.

As C is a simple, closed curve in $\mathbb{R}^2$, it encloses some region D such that $\partial D = C$.

We may apply Green's Theorem, where we know $\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$.

So

$$\oint_C f(x)\,dx + g(y)\,dy = \iint_D \left(\frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y}\right) dx\,dy$$

$$= \iint_D (0 - 0)\,dx\,dy$$

$$= 0.$$

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ \ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ \ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ \ y = \rho\sin\varphi\sin\theta,\ \ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \ \theta = \theta,\ \ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(x(u,v,w),y(u,v,w),z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.