MULTIVARIABLE CALCULUS
EXAM 3
FALL 2014

Name: Solution

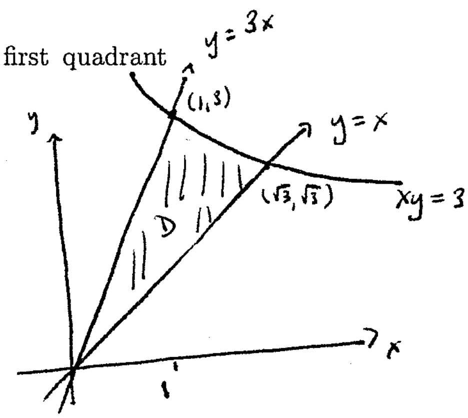
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Total: 70 points

Avg: 61.3 points - wow!

Directions: Complete all problems. Each problem is worth 10 points. Justify all answers/solutions. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices. The last two pages contains formulas. Best of luck.

(1) Evaluate $\int\int_D (x^2 + y^2) dA$, where D is the region in the first quadrant bounded by $y = x, y = 3x$ and $xy = 3$. (Hint: two pieces.)



We begin by sketching the region D

We break D into two regions: D_1, the portion of D to the left of $x=1$, and D_2, the portion of D to the right of $x=1$.

Fubini's Theorem applies.

So

$$\iint_D (x^2+y^2)\,dA = \iint_{D_1} (x^2+y^2)\,dA + \iint_{D_2} (x^2+y^2)\,dA$$

$$= \int_0^1 \int_x^{3x} (x^2+y^2)\,dy\,dx + \int_1^{\sqrt{3}} \int_x^{3/x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 yx^2 + \frac{y^3}{3}\Big|_x^{3x}\,dx + \int_1^{\sqrt{3}} yx^2 + \frac{y^3}{3}\Big|_x^{3/x}\,dx$$

$$= \int_0^1 3x^3 + \frac{27x^3}{3} - \left(x^3 + \frac{x^3}{3}\right)dx + \int_1^{\sqrt{3}} 3x + \frac{9}{x^3} - \left(x^3 + \frac{x^3}{3}\right)dx$$

Date: December 10, 2014.

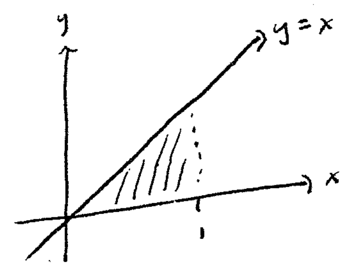
$$= \int_0^1 \frac{32}{3}x^3\,dx + \int_1^{\sqrt{3}} 3x + \frac{9}{x^3} - \frac{4}{3}x^3\,dx$$

$$= \frac{32}{3}\frac{x^4}{4}\Big|_0^1 + \left(\frac{3}{2}x^2 - \frac{9}{2x^2} - \frac{1}{3}x^4\Big|_1^{\sqrt{3}}\right)$$

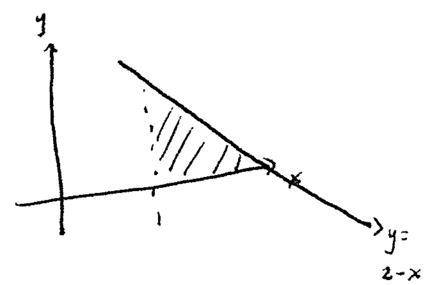
(2) Rewrite the given sum of iterated integrals as a single iterated integral by reversing the order of integration, and evaluate.

$$\int_0^1 \int_0^x \sin(x) dy\, dx + \int_1^2 \int_0^{2-x} \sin(x) dy\, dx$$

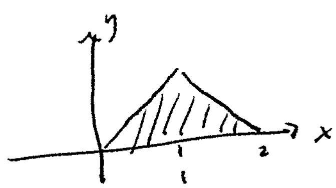
The first integral has the following region of integration:



The second integral has the following region of integration



Together the region is:



As a single integral with order of integration first x, then y is:

$$\int_{y=0}^{y=1}\int_{x=y}^{x=2-y} \sin(x)\, dx\, dy = \int_0^1 -\cos x \Big|_y^{2-y} dy$$

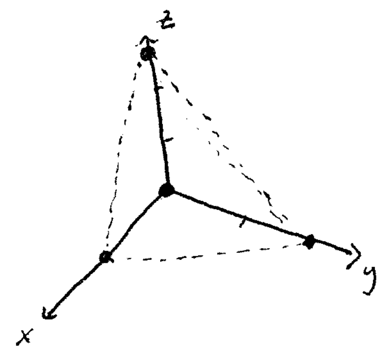
$$= \int_0^1 -\cos(2-y) + \cos y \; dy$$

$$= + \sin(2-y) + \sin y \Big|_0^1 = \sin 1 + \sin 1 - (\sin 2 + \sin 0)$$

$$= 2\sin 1 - \sin 2$$

(3) Integrate the given function over the indicated region W: $f(x,y,z) = 1 - z^2$; W is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$. } should only have asked for the set-up.

Let us sketch the region W.



We find an equation of the plane containing the latter 3 points: $P_0 = (1,0,0)$
$P_1 = (0,2,0)$
$P_2 = (0,0,3)$

$\overrightarrow{P_0P_1} = (-1,2,0)$ and $\overrightarrow{P_0P_2} = (-1,0,3)$ both lie in this plane. The normal vector to the plane may be calculated from their cross-product: $\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix}$

$= 6\vec{i} + 3\vec{j} + 2\vec{k}$

The equation for the plane is given $\vec{n} \cdot \overrightarrow{P_0P} = 0$

Thus, $(6\vec{i} + 3\vec{j} + 2\vec{k}) \cdot ((x-1)\vec{i} + y\vec{j} + z\vec{k}) = 0$

$6(x-1) + 3y + 2z = 0 \quad \Rightarrow \quad z = -\frac{3}{2}y - 3x + 3$

is an equation for the plane.

Thus, our integral is:

$$\int_0^1 \int_0^{2-2x} \int_0^{-3/2\,y - 3x + 3} (1 - z^2)\, dz\, dy\, dx$$

The line containing $(1,0,0)$ and $(0,2,0)$ is given by:
$y = -2(x-1) = -2x + 2$

We now integrate. But, geez, it's a mess to do so.

Full marks awarded for proper set-up. Final answer is 1/10, but no one wants to read the details

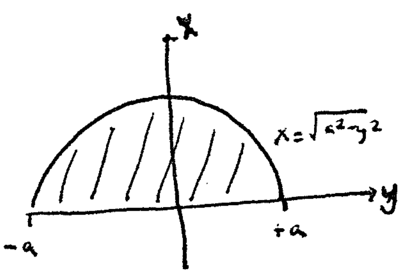
(4) Transform the given integral in Cartesian coordinates to one in polar coordinates and evaluate the polar integral.

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} e^{x^2+y^2}\,dx\,dy$$

Recall our change of coordinates is given by $r^2 = x^2+y^2$ and $\tan\theta = y/x$

Sketch the region.

$(x,y) = (r\cos\theta, r\sin\theta)$, w/ Jacobian equal to r.



$x = \sqrt{a^2-y^2} \Rightarrow x^2 = a^2-y^2$

$x^2+y^2 = a^2$

The integral equals

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{a} e^{r^2}\, r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2}e^{r^2}\Big|_0^a\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{e^{a^2}}{2} - \frac{1}{2}\, d\theta = \left(\frac{e^{a^2}}{2} - \frac{1}{2}\right)\theta\Big|_{-\pi/2}^{\pi/2}$$

$$= \pi\left(\frac{e^{a^2}}{2} - \frac{1}{2}\right)$$

(5) Calculate $\int_{\mathbf{x}} f\, ds$, where $f(x,y,z) = 3x + xy + z^3$, $\mathbf{x}(t) = (\cos(4t), \sin(4t), 3t)$, $0 \leq t \leq 2\pi$.

Recall the definition of the scalar line integral of f along a C^1 path $\vec{x}$:

$$\int_{\vec{x}} f\, ds = \int_a^b f(\vec{x}(t))\, \|\vec{x}\,'(t)\|\, dt.$$

Note $\vec{x}\,'(t) = (-4\sin 4t,\ 4\cos(4t),\ 3)$

so that $\|\vec{x}\,'(t)\| = \sqrt{16\sin^2 4t + 16\cos^2 4t + 9}$

$= \sqrt{16+9} = 5$

Thus,

$$\int_{\vec{x}} f\, ds = 5\int_0^{2\pi} 3\cos 4t + \cos 4t \sin 4t + 27t^3\, dt$$, using a double-angle formula

$$= 5\int_0^{2\pi} 3\cos 4t + \frac{1}{2}\sin 8t + 27t^3\, dt$$

$$= 5\left(\frac{3}{4}\sin 4t - \frac{1}{16}\cos 8t + \frac{27}{4}t^4 \Big|_0^{2\pi} \right)$$

$$= 5\left(\left[\frac{3}{4}\sin 8\pi - \frac{1}{16}\cos 16\pi + \frac{27}{4}16\pi^4\right] - \left[\frac{3}{4}\sin 0 - \frac{1}{16}\cos 0 + 0\right] \right)$$

$$= 5 \cdot 27 \cdot 4 \cdot \pi^4 = 540\pi^4$$

(6) Find $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j}, \mathbf{x}(t) = (2t+1, t+2),\ 0 \le t \le 1$.

The vector line integral of $\vec{F}$ along $\vec{x}$ is given by $\int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt$; it represents work done by $\vec{F}$

So,

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_0^1 (2t+1, t+2) \cdot (2,1)\, dt$$

$$= \int_0^1 4t+2 + t+2 \, dt = \int_0^1 5t+4 \, dt$$

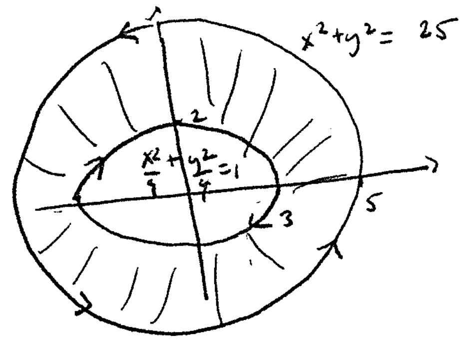
$$= \frac{5t^2}{2} + 4t \Big|_0^1$$

$$= \frac{5}{2} + 4 = \frac{13}{2}.$$

You really needed to use it, i.e. it's the point of the question.

(7) Use Green's theorem to find the area between the ellipse $x^2/9 + y^2/4 = 1$ and the circle $x^2 + y^2 = 25$. (Hint: a parametrization for the ellipse is $(3\cos(t), -2\sin(t))$.)

First, let's sketch the area:



The area we seek is that of the shaded region D. Its boundary consists of two simple, closed C^1 curves. We must orient them so that D is on the left as we traverse, as indicated by arrows in the figure. This has been done already for the ellipse. For the circle, the parametrization is $\vec{x}_1(t) = (5\cos t, 5\sin t)$, w/ $0 \le t \le 2\pi$.

Now, the area of D equals $\iint_D dA$, which by Green's Theorem (see page 430) equals $\frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy$.

This vector line integral is computed in two parts since ∂D consists of two pieces.

$$\underbrace{\frac{1}{2}\int_0^{2\pi} -5\sin t \cdot -5\sin t + 5\cos t \cdot 5\cos t \, dt}_{\text{Circle portion}} + \underbrace{\frac{1}{2}\int_0^{2\pi} 2\sin t \cdot -3\sin t + 3\cos t \cdot -2\cos t \, dt}_{\text{ellipse portion.}}$$

$$= \frac{1}{2}\int_0^{2\pi} 19\sin^2 t + 19\cos^2 t \, dt = \frac{19}{2}\int_0^{2\pi} dt = \frac{19}{2} t \Big|_0^{2\pi} = \frac{19}{2} \cdot 2\pi = 19\pi.$$

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2+y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.