MULTIVARIABLE CALCULUS
EXAM 3
FALL 2013

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Electronic devices, books, and notes are not permitted. Please turn off cell phones and other devices. The last two pages contains formulas. Best of luck.

(1) [10 points] Calculate the given iterated integral and indicate of what region in $\mathbb{R}^3$ it may be considered to represent the volume.

$$\int_{-2}^{3}\int_{0}^{1} |x| sin(\pi y)\ dy\ dx$$

$$\int_{-2}^{3}\int_{0}^{1} |x| \sin \pi y \, dy\, dx = \int_{-2}^{3} -\frac{|x|}{\pi}\cos(\pi y)\Big|_0^1 dx$$

$$= \int_{-2}^{3} -\frac{|x|}{\pi}\left(1-(-1)\right) dx$$

$$= \frac{2}{\pi}\int_{-2}^{3} |x|\, dx$$

$$= \frac{2}{\pi}\left(\int_{-2}^{0} -x\, dx + \int_{0}^{3} x\, dx\right)$$

$$= \frac{2}{\pi}\left(\frac{-x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{3}\right)$$

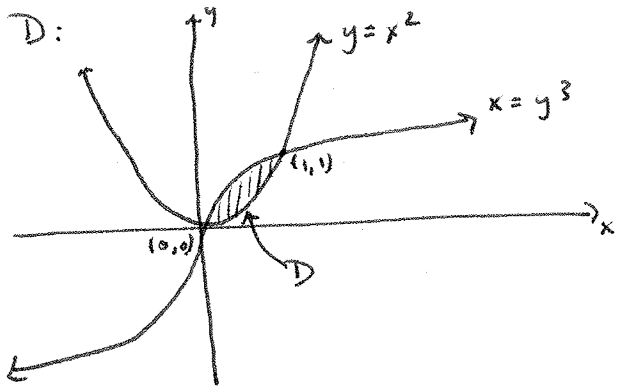
$$= \frac{2}{\pi}\left[\left(0-(-2)\right) + \left(\frac{9}{2}-0\right)\right] = \frac{13}{\pi}$$

The region in $\mathbb{R}^3$ is the volume enclosed by $z = |x| \sin(\pi y)$, the planes $y=0$, $y=1$, $x=-2$, $x=3$ and xy-plane.

Date: December 10, 2013.

(2) [10 points] Let D be the region bounded by $x = y^3$ and $y = x^2$. We are interested in computing $\iint_D xy\ dA$. Give an equivalent iterated integral where you treat D as a Type I elementary region. Then do the same when you treat D as a Type II elementary region. Finally, compute a numerical value for one of these iterated integrals.

Let us first sketch the region D:



As a Type I elementary region, we have:

$$\iint_D xy\ dA = \int_0^1 \int_{x^2}^{\sqrt[3]{x}} xy \quad dy\, dx$$

As a Type II elementary region, we have

$$\iint_D xy\ dA = \int_0^1 \int_{y^3}^{\sqrt{y}} xy \quad dx\, dy$$

We evaluate the latter as $\displaystyle\int_0^1 \frac{x^2}{2} y \Big|_{y^3}^{\sqrt{y}} \ dy$

$$= \frac{1}{2}\int_0^1 y \cdot y - y^6 \cdot y \ dy = \frac{1}{2}\int_0^1 y^2 - y^7\, dy$$

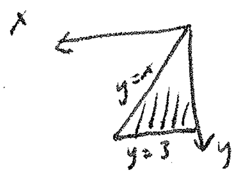
$$= \frac{1}{2}\left(\frac{y^3}{3} - \frac{y^8}{8} \Big|_0^1 \right) = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{8}\right) = \frac{5}{48}$$

(3) [10 points] Show how to set up the following triple integral as an iterated integral in one-half of the total number of ways possible, but do NOT evaluate the integral. Find $\iiint_W z\, dV$, where W is the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $y = x, x = 0$, and $z = 0$. (A Maple sketch of W is included with the exam packet.)

I will demonstrate all 6 possible ways, which is the maximum number of ways possible. So you needed to show 3 ways. For each type of region we demonstrate/draw the projection of W onto the appropriate plane, and give the corresponding iterated integrals.

Type I

Projection onto xy-plane

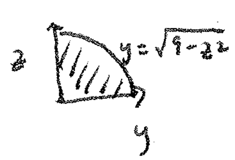


$$\int_0^3 \int_0^y \int_0^{\sqrt{9-y^2}} z \; dz\, dx\, dy$$

$$\int_0^3 \int_x^3 \int_0^{\sqrt{9-y^2}} z \; dz\, dy\, dx$$

Type II

projection onto yz-plane

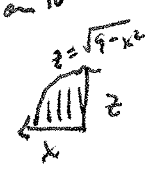


$$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^y z \; dx\, dy\, dz$$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^y z \; dx\, dz\, dy$$

Type III

projection onto xz-plane



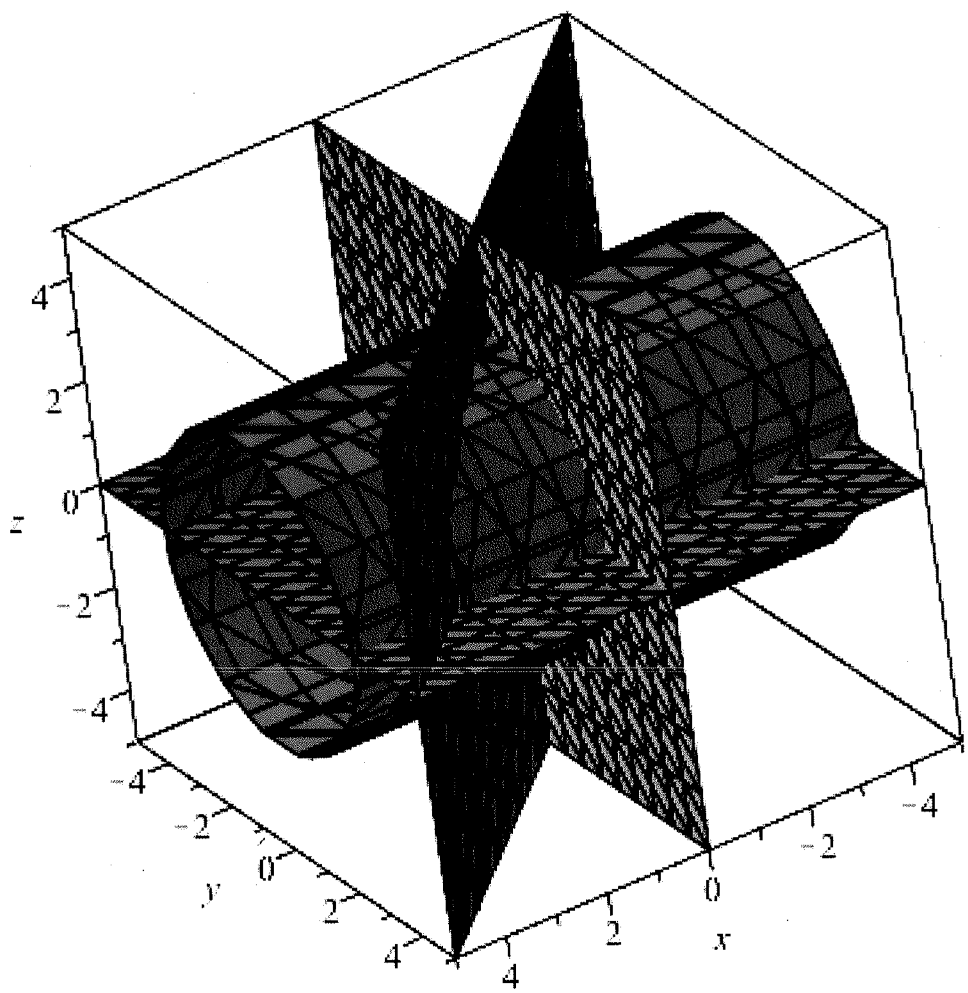
$$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_x^{\sqrt{9-z^2}} z \; dy\, dx\, dz$$

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_x^{\sqrt{9-z^2}} z \; dy\, dz\, dx$$

with(*plots*);

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] **(1)**

implicitplot3d$([y=x, x=0, z=0, y^2+z^2=9], x=-5..5, y=-5..5, z=-5..5,$ *color* = [*blue, green, purple, red*]);



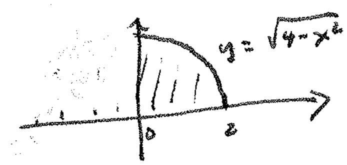
(4) [10 points] Transform the given integral in Cartesian coordinates to one in polar coordinates and evaluate the polar integral. Give a sketch of D and D^*. **Also,** find the area of D and the area of D^* and state the relation between these two areas.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} dy\, dx$$

As suggested we make the following change of variables. $x = r\cos\theta$, $y = r\sin\theta$. We recall that the Jacobian for this change of variables is $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{bmatrix} = r$.

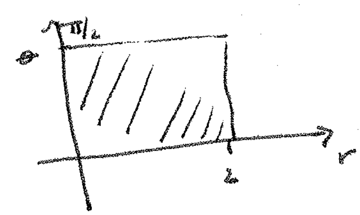
Note that D is given here:

$D = \{(x,y) \mid 0 \le y \le \sqrt{4-x^2},\ 0 \le x \le 2\}$



And D^* is given here

$D^* = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$



So the integral becomes

$$\int_0^{\pi/2}\int_0^2 r\, dr\, d\theta = \int_0^{\pi/2} 2\, d\theta = \pi.$$

And note that as our original integrand is 1, this represents the area of both D and D^*.

This matches what we know from basic geometry:

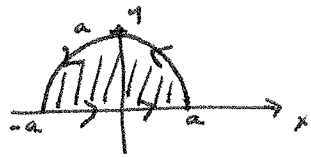
The area of D is $\frac{1}{4}\pi(2^2) = \pi$.

The area of D^* is $2 \cdot \frac{\pi}{2} = \pi$.

The areas of D and D^* are the same. This may seem puzzling in light of the relationship given in Proposition 5.1 but the transformation is not one-to-one so the Proposition does not apply. (See Example 7 on page 354 for a discussion on this transformation.)

(5) [10 points] Verify Green's theorem for the vector field $\mathbf{F} = 2y\mathbf{i} + x\mathbf{j}$, and D is the semicircular region $x^2 + y^2 \leq a^2$ and $y \geq 0$ by calculating both $\oint_{\partial D} M\,dx + N\,dy$ and $\int\int_D (N_x - M_y)\,dA$.

First we sketch D:



We may parametrize ∂D in the following manner:

$$x_1(t) = (a\cos t,\ a\sin t)\ ,\quad 0 \leq t \leq \pi$$

$$x_2(t) = (-a + 2at,\ 0)\ ,\quad 0 \leq t \leq 1.$$

Many people forgot this piece. Why?

This allows us to compute $\oint_{\partial D} M\,dx + N\,dy$ as

$$\int_0^{\pi} (2a\sin t,\ a\cos t)\cdot(-a\sin t,\ a\cos t)\,dt\ +$$

$$\int_0^1 (0,\ -a+2at)\cdot(2a,0)\,dt$$

$$= \int_0^{\pi} -2a^2\sin^2 t + a^2\cos^2 t\,dt + 0$$

$$= a^2\int_0^{\pi} -2\sin^2 t + \cos^2 t\,dt = a^2\int_0^{\pi} -2(1-\cos^2 t) + \cos^2 t\,dt$$

$$= a^2\int_0^{\pi} -2 + 3\cos^2 t\,dt$$

$$= a^2\int_0^{\pi} -2 + 3\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= a^2\left[-2t + \frac{3}{2}t + \frac{3}{2}\cdot\frac{1}{2}\sin 2t\right]_0^{\pi}$$

$$= -\frac{\pi}{2}a^2$$

We now compute $\iint_D N_x - M_y\,dA$. This equals $\iint_D (1-2)\,dA = \iint_D -1\,dA = -\iint_D dA$.

So we are finding negative of the area of region D. Thus, $-\frac{1}{2}\pi a^2$.

This verifys Green's Theorem since the two values are equal.

(6) [5 points] Calculate the following difference of two scalar line integrals, $\int_{\mathbf{x}} f\,ds - \int_{\mathbf{y}} g\,ds$, where $f(x,y,z) = x+y+z$ and $g(x,y,z) = z+y+x$, and

$$\mathbf{x}(t) = \begin{cases} (2t,0,0) & : \text{if } 0 \le t \le 1 \\ (2,3t-3,0) & : \text{if } 1 \le t \le 2 \\ (2,3,2t-4) & : \text{if } 2 \le t \le 3 \end{cases}$$

$$\mathbf{y}(t) = \begin{cases} (4t,0,0) & : \text{if } 0 \le t \le 1/2 \\ (2,6t-3,0) & : \text{if } 1/2 \le t \le 1 \\ (2,3,\frac{1}{50}t - \frac{1}{50}) & : \text{if } 1 \le t \le 101 \end{cases}$$

You might have done a bunch of computations, but I wanted you to notice that $f = g$ and that $\vec{y}(t)$ is a reparameterization of $\vec{x}(t)$. We can then take advantage of Theorem 1.4 of Section 6.1, which says that the scalar line integral is independent of the parametrization. Thus, we can conclude $\int_{\vec{x}} f\,ds - \int_{\vec{y}} g\,ds$ equals zero!

Change of coordinates

Cylindrical to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta, z = z$$

Cartesian to cylindrical:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

Spherical to Cartesian:

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$$

Cartesian to spherical:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan(\varphi) = \sqrt{x^2 + y^2}/z,\ \tan(\theta) = \frac{y}{x}$$

Spherical to cylindrical:

$$r = \rho\sin(\varphi),\ \theta = \theta,\ z = \rho\cos(\varphi)$$

Cylindrical to spherical:

$$\rho^2 = r^2 + z^2,\ \tan(\varphi) = r/z,\ \theta = \theta$$

Change of variables in triple integrals:

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))|\frac{\partial(x,y,z)}{\partial(u,v,w)}|dudvdw$$

Volume elements:

$$dV = dx\ dy\ dz\ Cartesian$$
$$dV = r\ dr\ d\theta\ dz\ Cylindrical$$
$$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\ spherical$$
$$dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw\ general$$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Pythagorean and reciprocal identities

- If you don't know these, then get a tattoo.