MULTIVARIABLE CALCULUS
EXAM 2
SPRING 2024

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Total 60 points

Average 54 points

Directions: Complete all problems. Justify all answers/solutions. Each problem is worth 10 points. Calculators/notes/texts/cell-phones are not permitted – the only permitted item is a writing utensil. Best of luck.

(1) The position vector of an object moving in a plane is given by $\mathbf{x}(t) = t^3\mathbf{i} + t^2\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$. Give an appropriate sketch of the path for $t \geq 0$. Then find an equation for the line tangent to $\mathbf{x}$ at $\mathbf{x}(1)$.

The velocity is given by $\vec{v}(t) = 3t^2\vec{i} + 2t\vec{j}$

and acceleration $\vec{a}(t) = 6t\vec{i} + 2\vec{j}$

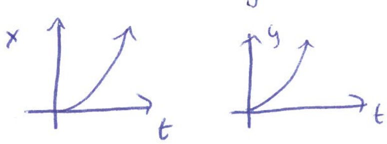
At $t=1$, the velocity is $\vec{v}(1) = 3\vec{i} + 2\vec{j}$

the speed is $s(1) = \sqrt{9+4} = \sqrt{13}$

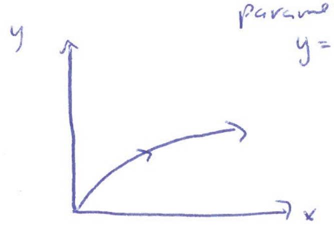
the acceleration is $\vec{a}(1) = 6\vec{i} + 2\vec{j}$

Sketch

$x(t) = t^3$ $\quad$ $y(t) = t^2$



$\Downarrow$ eliminate parameter $y = x^{2/3}$



Tangent line at $t=1$:

$$\begin{aligned} \ell(t) &= \vec{x}(1) + \vec{v}(1)(t-1) \\ &= (\vec{i} + \vec{j}) + (3\vec{i} + 2\vec{j})(t-1) \\ &= (3t-2)\vec{i} + (2t-1)\vec{j} \end{aligned}$$

Date: April 18, 2024.

(2) Set up the correct definite integral for computing the length of the given path. You do not need to solve it.

$$\mathbf{x}(t) = (\ln(\cos(t)), \cos(t), \sin(t)),\ \frac{\pi}{6} \leq t \leq \frac{\pi}{3}$$

length of the path is given by

$$L(\vec{x}) = \int_a^b \|\vec{x}\,'(t)\|\,dt.$$

Here we have $\vec{x}\,'(t) = \left(\frac{1}{\cos t}(-\sin t), -\sin t, \cos t\right) = (-\tan t, -\sin t, \cos t)$

Thus, $\|\vec{x}\,'(t)\| = \sqrt{\tan^2 t + \sin^2 t + \cos^2 t}$

$= \sqrt{\tan^2 t + 1}$

Thus, $$L(\vec{x}) = \int_{\pi/6}^{\pi/3} \sqrt{\tan^2 t + 1}\,dt.$$

(3) Calculate the flow line $\mathbf{x}(t)$ of the vector field $\mathbf{F}(x,y) = (x^2, y)$ at the point $\mathbf{x}(1) = (1, e)$. Sketch this flow line starting at $t = 1$. Calculate the divergence of $\mathbf{F}$ at this point.

By Definition 3.2 on page 225, the flow line satisfies $\vec{x}\,'(t) = F(\vec{x}(t))$

Thus, if $\vec{x}(t) = (x(t), y(t))$, then we must have

(1) $\dfrac{dx}{dt} = x^2$ and (2) $\dfrac{dy}{dt} = y$.

These are separable differential equations which we now solve.

(1) $\displaystyle\int \frac{dx}{x^2} = \int dt$

$-\dfrac{1}{x} = t + c$

$\Rightarrow x = \dfrac{-1}{t + C}$

As $x(1) = 1$, we have

$1 = \dfrac{-1}{1 + C}$

$\Rightarrow C = -2$

$x = \dfrac{-1}{t - 2}$

(2) $\displaystyle\int \frac{dy}{y} = \int dt$

$\ln|y| = t + C$

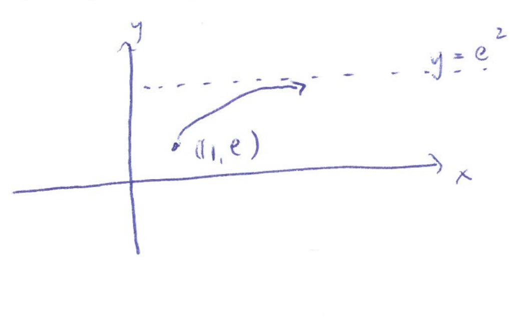
$y = Ae^t$

As $y(1) = e$,

$e = Ae^1 \Rightarrow A = 1$

So, $y = e^t$

Flow line sketch



So, $\vec{x}(t) = \left(\dfrac{-1}{t-2}, e^t \right)$

The divergence at $t = 1$:

$\operatorname{div} F = \vec{\nabla} \cdot \vec{F} = 2x + 1$

at $(1, e)$ $\vec{\nabla} \cdot \vec{F} = (2 \cdot 1 + 1) = 3$.

(4) Find the first- and second-order Taylor polynomials for the function

$$f(x, y) = e^x \sin(y)$$

at the point $\mathbf{a} = (1, \frac{\pi}{2})$.

The first-order taylor polynomial (as given by Theorem 1.3 on page 248) is:

$$P_1(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

$$f(1, \tfrac{\pi}{2}) = e^1 \sin(\tfrac{\pi}{2}) = e$$

$$Df(x,y) = \begin{bmatrix} e^x \sin y & e^x \cos y \end{bmatrix} \qquad Df(1, \tfrac{\pi}{2}) = \begin{bmatrix} e & 0 \end{bmatrix}$$

So, $p_1(\vec{x}) = e + \begin{bmatrix} e & 0 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - \frac{\pi}{2} \end{bmatrix} = e + e(x-1) = ex$

Now $p_2(\vec{x}) = p_1(\vec{x}) + \frac{1}{2}(\vec{x} - \vec{a})^T Hf(\vec{a})(\vec{x} - \vec{a})$

$$Hf = \begin{bmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{bmatrix} \qquad Hf(1, \tfrac{\pi}{2}) = \begin{bmatrix} e & 0 \\ 0 & -e \end{bmatrix}$$

So, $p_2(\vec{x}) = e^x + \frac{1}{2}\begin{bmatrix} x-1 & y - \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & -e \end{bmatrix} \begin{bmatrix} x - 1 \\ y - \frac{\pi}{2} \end{bmatrix}$

$$= e^x + \frac{1}{2}\begin{bmatrix} x-1 & y - \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} e(x-1) \\ -e(y - \pi/2) \end{bmatrix}$$

$$= e^x + \frac{1}{2}\left(e(x-1)^2 - e(y - \tfrac{\pi}{2})^2\right)$$

$$= e^x + \frac{e}{2}\left((x-1)^2 - (y - \tfrac{\pi}{2})^2\right)$$

(5) Use the second derivative test for functions of two variables to determine the nature of the critical points of the following function which I'm telling you are $(0,0)$ and $(2,0)$.

$$f(x,y) = e^{-x}(x^2+3y^2) = x^2e^{-x} + 3e^{-x}y^2$$

Let us find the Hessian:

$$Df = \begin{bmatrix} -e^{-x}x^2 + 2xe^{-x} - 3e^{-x}y^2 & 6e^{-x}y \end{bmatrix}$$

$$Hf = \begin{bmatrix} -2e^{-x}x + e^{-x}x^2 - 2xe^{-x} + 2e^{-x} + 3e^{-x}y^2 & -6e^{-x}y \\ -6e^{-x}y & 6e^{-x} \end{bmatrix}$$

At $(0,0)$

$$Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

Thus $d_1 = 2$ and $d_2 = 12$

Both positive.

So $(0,0)$ is local min.

At $(2,0)$

$$Hf(2,0) = \begin{bmatrix} -2e^{-2} & 0 \\ 0 & 6e^{-2} \end{bmatrix}$$

$d_1 = -2e^{-2}$, a negative

and $d_2 = -12e^{-4}$, a negative.

So this is a saddle point.

(6) Use the method of Lagrange multipliers to find the extreme values of the function $f(x, y, \cancel{z}) = x^2 + 2y^2$ subject to the condition $x^2 + y^2 = 1$.

Write the system of equations here.

Let $g(x,y) = x^2 + y^2$.

The system is given by $\nabla f = \lambda \nabla g$, $g(x,y) = 1$.

$$\begin{bmatrix} 2x & 4y \end{bmatrix} = \lambda \begin{bmatrix} 2x & 2y \end{bmatrix}$$

So,

$$2x = \lambda 2x$$
$$4y = \lambda 2y$$
$$x^2 + y^2 = 1$$

Note that the first equation gives $x = 0$ or $\lambda = 1$. Use this hint to find the critical points.

If $x = 0$, then $y^2 = 1$ so $y = \pm 1$. Thus, $(x,y) = (0, \pm 1)$

If $\lambda = 1$, then $4y = 2y$ so $y = 0$ and $x^2 = 1$

Thus $(x,y) = (\pm 1, 0)$.

Evaluate these critical points on the function to determine which correspond to maximum and which to minimum.

$f(0,1) = 2$
$f(0,-1) = 2$ } correspond to maximum

$f(1,0) = 1$
$f(-1,0) = 1$ } correspond to minimum

Say why!