MULTIVARIABLE CALCULUS
EXAM 2
SPRING 2021

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid.

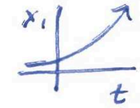
Avg $\frac{64}{70}$ whoa!

Directions: Complete all problems. Justify all answers/solutions. Each problem is worth 10 points. Calculators/notes/texts/cell-phones are not permitted – the only permitted item is a writing utensil. Best of luck.

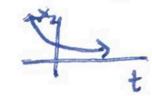
(1) Sketch the image of the following path, using arrows to indicate the direction in which the parameter increases:

$$\mathbf{x}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$$

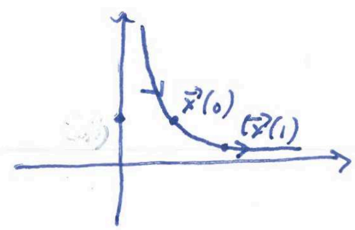
We first sketch by component $x_1(t) = e^t$



$x_2(t) = e^{-t}$



So, as $t \to \infty$, x_1 gets large and x_2 gets small.



Now find an equation for the line tangent to this path at $t = 1$.

At $t = 1$, $\vec{x}(1) = e\,\vec{i} + \frac{1}{e}\vec{j}$, $\vec{x}(1) = (e, \frac{1}{e})$

$\nabla\vec{x}(t) = (e^t, -e^{-t}) = \vec{v}(t)$

$\vec{v}(1) = (e, -\frac{1}{e})$ Thus the tangent line is

$\vec{\ell}(t) = (e, \frac{1}{e}) + (t-1)(e, \frac{-1}{e})$

Date: April 22, 2021.

(2) Consider the curve $g(t) = (t, \frac{2}{3}t^{3/2})$. Give the arc length $s(t)$ in terms of the parameter t measured from 0. Then solve for t in terms of s. Next give the function with uniform speed 1 which traces the same curve.

We know that arc length is given by

$$S(t) = \int_0^t \|g'(\tau)\| \, d\tau$$

Thus, as $g'(\tau) = (1, \tau^{1/2})$ and so $\|g'(\tau)\| = \sqrt{1+\tau}$

$$s(t) = \int_0^t \sqrt{1+\tau} \, d\tau$$

$$= \frac{2}{3}(1+\tau)^{3/2} \Big|_0^t = \frac{2}{3}(1+t)^{3/2} - \frac{2}{3}$$

So, $s = \frac{2}{3}(1+t)^{3/2} - \frac{2}{3}$

Solving for t in terms of s, we get

$$s + \frac{2}{3} = \frac{2}{3}(1+t)^{3/2}$$

$$1 + \frac{3}{2}s = (1+t)^{3/2} \Rightarrow (1 + 3/_2 s)^{2/3} - 1 = t$$

So, let $h(s) = g\left((1+\frac{3}{2}s)^{2/3} - 1\right)$

and we have $\|h'(s)\| = 1$.

(3) Describe to me in words the gradient field of the following function,

$$g(x,y,z) = \frac{1}{4}(x^2 + y^2 + z^2).$$

Your response should include use of the words "direction" and "magnitude".

Perhaps I should have asked a more pointed question?

The gradient of g is given by

$$\nabla g(x,y,z) = \left(\tfrac{1}{2}x, \tfrac{1}{2}y, \tfrac{1}{2}z\right) = \tfrac{1}{2}(x,y,z)$$

So at each point $(x,y,z) \in \mathbb{R}^3$ sits a vector that points away from the origin, that is, its direction is "on" the line b/t the origin and the point itself.

The magnitude is $\| \tfrac{1}{2}(x,y,z) \| = \tfrac{1}{2} \| (x,y,z) \|$, meaning that points far from the origin have vectors associated w/ them that are large in magnitude and growing larger as we move away from the origin.

(4) If $\mathbf{F}(x,y,z) = (y, z^2, x^3)$, then describe the vectors of the vector field $curl\mathbf{F}$ that lie along the y-axis (i.e. where $x = z = 0$). Your response should include use of the words "direction" and "magnitude".

We may compute the curl F as follows:

$$\text{Curl } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z^2 & x^3 \end{vmatrix}$$

$$= \left(\frac{\partial x^3}{\partial y} - \frac{\partial z^2}{\partial z} \right) \vec{i} - \left(\frac{\partial x^3}{\partial x} - \frac{\partial y}{\partial z} \right) j + \left(\frac{\partial z^2}{\partial x} - \frac{\partial y}{\partial y} \right) \vec{k}$$

$$= -2z\,\vec{i} - 3x^2\,\vec{j} - 1\,\vec{k}$$

So, when $x = z = 0$, we are left with $-1\vec{k}$.

That is, the vectors of the vector field where $x = z = 0$ all have magnitude 1 and all point in the direction of the negative z-axis.

(5) Find the first- and second-order Taylor polynomials for the function

$$f(x,y) = ye^{3x} + e^{2y}$$

at the point $\mathbf{a} = (0,2)$.

$f(\vec{a}) = 2e^0 + e^4 = e^4 + 2$

$\frac{\partial f}{\partial x} = 3ye^{3x} = 6e^0 = 6$

$Df(\vec{a}) = [6 \quad 1+2e^4]$

$\frac{\partial f}{\partial y} = e^{3x} + 2e^{2y} = 1+2e^4$

$P_1(\vec{x}) = e^4+2+(6)(x-0)+(1+2e^4)(y-2)$

$P_1(\vec{x}) = e^4+2+6x+(1+2e^4)(y-2)$

$Hf = \begin{bmatrix} 9ye^{3x} & 3e^{3x} \\ 3e^{3x} & 4e^{2y} \end{bmatrix}$ @ $(0,2)$: $\begin{bmatrix} 18 & 3 \\ 3 & 4e^4 \end{bmatrix}$

$P_2(\vec{x}) = f(\vec{a}) + Df(\vec{a})\vec{h} + \frac{1}{2}\vec{h}^T Hf(\vec{a})\vec{h}$

$P_2(\vec{x}) = e^4+2+\begin{bmatrix} 6 & 1+2e^4 \end{bmatrix}\begin{bmatrix} x-0 \\ y-2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y-2 \end{bmatrix}\begin{bmatrix} 18 & 3 \\ 3 & 4e^4 \end{bmatrix}\begin{bmatrix} x \\ y-2 \end{bmatrix}$

$= e^4+2+6x+(1+2e^4)(y-2)+\frac{1}{2}\begin{bmatrix} 18x+3y-6 & 3x+4e^4(y-2) \end{bmatrix}\begin{bmatrix} x \\ y-2 \end{bmatrix}$

$= e^4+2+6x+(1+2e^4)(y-2)+\frac{1}{2}(18x^2+3xy-6x+(y-2)(3x+4e^4(y-2)))$

$= e^4+2+6x+y-2+2ye^4-4e^4+9x^2+\frac{3}{2}xy-3x+\frac{1}{2}((y-2)(3x+4ye^4-8e^4))$

$= e^4+6x+y+2ye^4-4e^4+9x^2+\frac{3}{2}xy-3x+\frac{1}{2}(3xy+4y^2e^4-8ye^4-6x$
$-8ye^4+16e^4)$

$= e^4+6x+y+2ye^4-4e^4+9x^2+\frac{3}{2}xy-3x+\frac{3}{2}xy+2y^2e^4-4ye^4-3x-4ye^4+8e^4$

$P_2(\vec{x}) = 5e^4+y-6ye^4+9x^2+3xy+2y^2e^4$

(6) Consider the function $f(x, y) = x^2 + y^2$ for points (x, y) in the square $U = \{(x, y) | -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Find the critical points of f. Determine the nature of these critical points. Does f have a global maximum on U? If yes, suggest where it might be. If no, explain why not.

We find the critical points by setting $\nabla f(x,y) = 0$.

So, we get $\nabla f(x,y) = (2x, 2y) = (0,0)$

$\Rightarrow x = 0, y = 0$ and so there is one critical point at $(0,0)$. We can avoid using the second derivative test by noting that $f(0,0) = 0$ and $f(x,y) > 0$ for any other point (x,y).

So, we see $(0,0)$ is a local minimum (in fact, a global minimum).

If f has a global maximum on U, then it must occur on the boundary of U.

If $x = 1$, then $f(1,y) = 1 + y^2$, which is largest at $y = \pm 1$

Similarly, if $x = -1$, or $y = 1$ or $y = -1$.

So maximums occur at the corners of the square U.

That is $(1,1), (1,-1), (-1,1), (-1,-1)$ are maximums w/ value 2.

and solve

(7) Set up (~~but do not solve~~) the Lagrange equations that would allow one to determine the maximum value of $f(x, y, z) = x - y + z$, subject to the condition $x^2 + y^2 + z^2 = 1$.

don't need to determine nature.

We wish to solve the Lagrange equations:

$$\nabla f = \lambda \nabla g$$

$$g(x,y,z) = 1 .$$

So, we get

$$\nabla f(x,y,z) = (1, -1, 1)$$

$$\nabla g(x,y,z) = (2x, 2y, 2z)$$

Thus

$$1 = \lambda 2x$$

$$-1 = \lambda 2y$$

$$1 = \lambda 2z$$

$$x^2 + y^2 + z^2 = 1 .$$