MULTIVARIABLE CALCULUS
EXAM 2
SPRING 2018

80 points total

average was 73 points - wow!

Name: Solutions
Honor Code Statement: I have neither given nor received unauthorized aid. C.F. Gauss.

Directions: Complete all problems. Justify all answers/solutions. Computational problems are worth 10 points each; others are worth 5 points each. Calculators/notes/texts/cell-phones are not permitted – the only permitted item is a writing utensil. Best of luck.

(1) Find an equation for the line tangent to the path $\mathbf{x}(t) = (\cos(e^t), 3 - t^2, t)$ at the value $t = 1$.

The point of tangency is given by $\vec{x}(1) = (\cos(e), 3-1, 1)$
$= (\cos(e), 2, 1)$

The velocity is given by $\vec{x}\,'(t) = (-e^t \sin(e^t), -2t, 1)$

At $t=1$, the velocity vector is $\vec{x}\,'(1) = (-e^1 \sin(e), -2, 1)$

The tangent line is the line thru $\vec{x}(1)$ that is parallel to $\vec{x}\,'(1)$.

It is

$$\ell(t) = (\cos(e), 2, 1) + (t-1)(-e\sin(e), -2, 1).$$
$$= (-te\sin(e) + e\sin(e) + \cos(e), -2t+4, t)$$

Date: April 19, 2018.

(2) Find the length of the path $\mathbf{x}(t) = (t^3, 3t^2, 6t)$ for $-1 \le t \le 2$.

As stated on page 204 of the text, the length of a C^1 path $\vec{x} : [a,b] \to \mathbb{R}^n$ is given the integral of the speed. That is,

$$L(\vec{x}) = \int_a^b \| \vec{x}\,'(t) \| \, dt.$$

So we find the speed $\| \vec{x}\,'(t) \| = \sqrt{(3t^2)^2 + (6t)^2 + 6^2}$

$$= \sqrt{9t^4 + 36t^2 + 36}$$

$$= \sqrt{9(t^4 + 4t^2 + 4)}$$

$$= \sqrt{9(t^2+2)^2}$$

$$= 3(t^2+2).$$

So,

$$L(\vec{x}) = \int_{-1}^{2} 3(t^2+2)\, dt$$

$$= \frac{3t^3}{3} + 3 \cdot 2 \cdot t \Big|_{-1}^{2}$$

$$= t^3 + 6t \Big|_{-1}^{2} = (8 + 12) - (-1 - 6) = 27.$$

This is valid since $\vec{x}(t)$ is indeed a C^1 path.

(3) Define *vector field.*

A vector field on $\mathbb{R}^n$ is

a mapping $F: X \subseteq \mathbb{R}^n \to \mathbb{R}^n$

(4) Some of these are scalar fields and some are vector fields, say what each is.

- grad$f$ - vector field
- div $\mathbf{F}$ - scalar field
- curl $\mathbf{F}$ - vector field
- grad$f\times$ div $\mathbf{F}$ - neither, one cannot make such a computation.

(5) Calculate the flow line $\mathbf{x}(t)$ of the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ at the point $\mathbf{x}(0) = (1, 2)$.

The flow line of a vector field F is a differentiable path $\vec{x}$ such that $\vec{x}\,'(t) = F(\vec{x}(t))$

If $\vec{x}(t) = (x, y)$ then $\vec{x}\,'(t) = F(x, y) = (x, -y)$.

(1) Then $\frac{dx}{dt} = x$. This is a separable differential equation.

$\int \frac{dx}{x} = \int dt \Rightarrow \ln|x| = t + C \Rightarrow x = Ce^{t}$

When $t=0$, $x=1$ and so $1 = Ce^{0} \Rightarrow C = 1$. $\Rightarrow x = e^{t}$.

(2) Then $\frac{dy}{dt} = -y$. This is a separable differential equation.

$\int \frac{dy}{y} = \int -dt \Rightarrow \ln|y| = -t + C \Rightarrow y = Ce^{-t}$

When $t=0$, $y = 2$ and so $2 = Ce^{-0} \Rightarrow C = 2 \Rightarrow y = 2e^{-t}$

So the equation of the flow line is

$$\vec{x}(t) = (e^{t}, 2e^{-t})$$

(6) Given $\mathbf{F}(x, y, z) = xyz\mathbf{i} - e^z \cos(x)\mathbf{j} + xy^2z^3\mathbf{k}$, determine $\nabla\cdot(\nabla \times \mathbf{F})$. Use words to say what it is that is being computed.

To compute $\nabla\cdot(\nabla\times\vec{F})$ we "work from the inside out".

So we first compute

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -e^z\cos x & xy^2z^3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial(xy^2z^3)}{\partial y} - \frac{\partial(-e^z\cos x)}{\partial z}\right) - \vec{j}\left(\frac{\partial(xy^2z^3)}{\partial x} - \frac{\partial}{\partial z}(xyz)\right)$$

$$+ \vec{k}\left(\frac{\partial(-e^z\cos x)}{\partial x} - \frac{\partial(xyz)}{\partial y}\right)$$

$$= i\left(2xyz^3 + e^z\cos x\right) - \vec{j}\left(y^2z^3 - xy\right) + \vec{k}\left(e^z\sin x - xz\right)$$

Then

$$\nabla\cdot(\nabla\times\vec{F}) = \frac{\partial}{\partial x}\left(2xyz^3 + e^z\cos x\right) + \frac{\partial}{\partial y}\left(y^2z^3 - xy\right) + \frac{\partial}{\partial z}\left(e^z\sin x - xz\right)$$

$$= 2yz^3 - e^z\sin x - 2yz^3 + x + e^z\sin x - x$$

$$= 0$$

This computes the divergence of the curl of $\vec{F}$.

(7) Find the first- and second-order Taylor polynomials for $f(x, y, z) = ye^{3x} + ze^{2y}$ at $\mathbf{a} = (0, 0, 2)$.

The first-order Taylor polynomial is given by $p_1(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$

the second-order Taylor polynomial is given by $p_2(x) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a}) + \frac{1}{2}(\vec{x} - \vec{a})^T Hf(\vec{a})(\vec{x} - \vec{a})$

So we must compute:

$$f(0,0,2) = 0 + 2e^0 = 2$$

$$f_x = 3ye^{3x} \qquad f_x(0,0,2) = 0$$

$$f_y = e^{3x} + 2ze^{2y} \quad \Longrightarrow \quad f_y(0,0,2) = 1 + 2\cdot 2 \cdot 1 = 5$$

$$f_z = 0 + e^{2y} \qquad f_z(0,0,2) = 0 + e^0 = 1$$

Thus, $p_1(\vec{x}) = 2 + (0, 5, 1) \cdot (x - 0, y - 0, z - 2)$

$$= 2 + 0 + 5y + z - 2 = 5y + z$$

Next,

$$f_{xx} = 9ye^{3x} \qquad f_{xx}(0,0,2) = 0$$

$$f_{xy} = f_{yx} = 3e^{3x} \qquad f_{xy}(0,0,2) = f_{yx}(0,0,2) = 3$$

$$f_{xz} = f_{zx} = 0 \quad \Longrightarrow \quad f_{xz}(0,0,2) = f_{zx}(0,0,2) = 0$$

$$f_{yy} = 0 + 4ze^{2y} \qquad f_{yy}(0,0,2) = 8$$

$$f_{yz} = f_{zy} = 2e^{2y} \qquad f_{yz}(0,0,2) = f_{zy}(0,0,2) = 2$$

$$f_{zz} = 0 \qquad f_{zz}(0,0,2) = 0$$

$$P_2(x,y,z) = (5y+z) + \frac{1}{2}\cdot 0\cdot(x-0)^2 + \frac{1}{2}\cdot 3\,(x-0)(y-0) + \frac{1}{2}\cdot 3\,(y-0)(x-0)$$

$$+ \frac{1}{2}\cdot 0\cdot(x-0)(z-2) + \frac{1}{2}\cdot 0\cdot(z-2)(x-0) + \frac{1}{2}\cdot 8\,(y-0)^2$$

$$+ \frac{1}{2}\cdot 2\,(y-0)(z-2) + \frac{1}{2}\cdot 2\,(z-2)(y-0) + \frac{1}{2}\cdot 0\cdot(z-0)^2$$

$$= 5y+z + 3xy + 4y^2 + 2y(z-2)$$

$$= y + z + 3xy + 4y^2 + 2yz$$

(8) Identify and determine the nature of the critical points of $f(x, y) = e^{-y}(x^2 - y^2)$.

By Theorem 2.2 of Section 4.2, if f has a local extremum at $\vec{a}$, then $Df(\vec{a}) = \vec{0}$. So we find the derivative and set to zero.

$$Df = [\, f_x \quad f_y \,] = [\, 2xe^{-y} \quad -e^{-y}(x^2-y^2) + e^{-y}(-2y) \,]$$

$$= [\, 2xe^{-y} \quad -e^{-y}(x^2-y^2+2y) \,]$$

Now $2xe^{-y} = 0 \iff x = 0$. Thus any critical point has $x=0$. Substituting $x=0$ into $e^{-y}(x^2-y^2+2y)$ we get $e^{-y}(-y^2+2y)$, and this equals 0 iff $-y^2+2y=0$ iff $-y(y-2)=0$ iff $y=0$ or $y=2$. Thus critical points are $(0,0)$ and $(0,2)$.

We now use the 2nd derivative test. We form the Hessian:

$$Hf(x,y) = \begin{bmatrix} 2e^{-y} & -2xe^{-y} \\ -2xe^{-y} & e^{-y}(x^2-y^2+2y) + -e^{-y}(-2y+2) \end{bmatrix}$$

Evaluate at each critical point

$$Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 0-1(2) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$Hf(0,2) = \begin{bmatrix} 2e^{-2} & 0 \\ 0 & e^{-2}\cdot 0 - e^{-2}\cdot -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2} & 0 \\ 0 & +2e^{-2} \end{bmatrix}$$

For $(0,0)$,

$d_1 = +2$ and $d_2 = -4$. Thus $(0,0)$ is a saddle point

For $(0,2)$

$d_1 = 2e^{-2} > 0$ and $d_2 = +4e^{-4} > 0$. Thus $(0,2)$ is a local minimum.

(9) A cylindrical metal can is to be manufactured from a fixed amount of sheet metal. Use the method of Lagrange multipliers to determine the ratio between the dimensions of the can with the largest capacity. **Please:** Set up the system of equations to solve; you need not solve it.

The volume of a cylinder is given by $V = \pi r^2 h$

The amount of sheet metal to be used is given by the surface area of a cylinder.

$$S = \underbrace{2\pi r^2}_{\text{top and botton}} + \underbrace{2\pi r h}_{\text{side}}.$$

We are told this is a fixed amount, say c.

So we wish to maximize $V = \pi r^2 h$ subject to the constraint $2\pi r^2 + 2\pi r h = c$.

We use the method of Lagrange Multipliers we form the vector equation $\nabla V = \lambda \nabla S$ and solve the system

$$2\pi r h = \lambda(4\pi r + 2\pi h)$$

$$\pi r^2 = \lambda(2\pi r)$$

$$2\pi r^2 + 2\pi r h = c$$

[Going beyond what was asked: the second equation implies $r = 2\lambda$, since r can't be zero. Plugging into first equation, we get $h = 2r$. That is, height equals diameter.]